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Chu I: cofree equivalences, dualities and $*$-autonomous categories

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We study three comonads derived from the comma construction. The induced coalgebras correspond to the three concepts displayed in the title of the paper. The comonad that yields the $*$-autonomous categories is, in essence, the Chu construction, which has recently awaken much interest in computer science. We describe its couniversal property. It is right adjoint to the inclusion of $*$-autonomous categories among autonomous categories, with lax structure-preserving morphisms. Moreover, this inclusion turns out to be comonadic: $*$-autonomous categories are exactly the Chu-coalgebras.

1. Introduction

The Chu construction was devised by Michael Barr and his student Po-Hsiang Chu as means to show that there are plenty of $*$-autonomous categories. It first appeared in Chu's master's thesis and in the appendix of Barr's book (Barr 1979). It differed in spirit from the methods pursued in the rest of the book, and looked a bit mysterious and *ad hoc*. Strangely enough, starting from a completely different background, Hyland and his student de Paiva (de Paiva 1989) arrived at a very similar structure, which they called "Dialectica" categories. While Barr and Chu worked with Banach spaces and Hopf algebras, Hyland and de Paiva were motivated by Gödel's "Dialectica" interpretation of constructive logic.

The importance of all this grew considerably when Seely (Seely 1989) noticed that $*$-autonomous categories provide models for linear logic. Barr (Barr 1990; Barr 1991) picked up the Chu construction as a general method of providing such models. He proposed a modification which, under certain conditions, guarantees not only the $*$-autonomous structure, but also linear exponentials. This modification and its advantages will be studied in part II of this paper.

The linear logic connection brought the Chu and the Dialectica categories on the scene of computer science and various computational interpretations started appearing (Lafont and Streicher 1991; Brown and Gurr 1991). The Chu *spaces*, obtained by applying the Chu construction to the category of sets, have been proposed by Pratt and his collaborators as a foundational structure for concurrency theory, capturing the duality of states and

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events (Gupta and Pratt 1993; Pratt 1993a; Pratt 1993b; Pratt 1994a; Pratt 1994b; Gupta 1994; Glabbeek and Plotkin 1995; Pratt 1995). They were shown to be remarkably rich and versatile, accomodating concrete faithful functors from arbitrary small concrete categories. However, no categorical universal property (Mac Lane 1971, ch. III) of the Chu construction has been established so far, no explanation in the style "it is the smallest (or the largest) category such that...". So there may still be grounds for arguing that it is an *ad hoc* structure, accidentally rich.

The purpose of the present paper is to establish a couniversal property of the Chu construction. Given an autonomous (i.e., closed symmetric monoidal) category with a chosen object, it produces a cofree one, such that the chosen object induces a duality. And an autonomous category with a dualizing object is, of course, $*$-autonomous.

While based on a rather trivial observation that the original presentation of the Chu construction (Chu 1979; Barr 1991) conceals a comma category, this result turns out to be not just technically demanding, but also conceptually evasive. (An earlier version led to a complaint that even if the Chu construction is not *ad hoc*, the category in which it is couniversal, still is.) In order to uncover the source and the real meaning of this couniversality, we are led to start two steps away from the Chu construction. In section 2, we describe a very general comonad, derived from the comma construction, and show how it extracts, among arbitrary functors, exactly the equivalences as coalgebras. The modifications needed for dealing with the contravariant functors are discussed in section 3. The comonad described there transforms arbitrary self-adjoint functors into a dualities. By adding the autonomous structure, we derive in section 4 a comonad on the 2-category of autonomous categories, each with a self-adjunction induced by a chosen object $\perp$. This is the Chu comonad. Its coalgebras are the $*$-autonomous categories, its homomorphisms — the structure preserving functors. The forgetful functor from these coalgebras turns out to be essentially an inclusion: each autonomous category with $\perp$ can be a coalgebra in at most one way, up to isomorphism. Being a Chu-coalgebra is thus a property of a given autonomous category with $\perp$, rather than additional structure. On the other hand, for any autonomous category with $\perp$, the Chu construction yields the cofree $*$-autonomous one. This is its couniversal property — originating from the couniversal property of the comma consturction.

But Barr and Chu did not set up their construction as a comma categories. Abstracting from the technique of dual pairs in functional analysis (Kelley, Nanmioka *et al.* 1963, ch. 5), they defined the objects of their category to be the triples $\langle A, B, A \otimes B \xrightarrow{\phi} \perp \rangle$, where A and B are arbitrary objects of an autonomous category $\mathcal{V}$, and $\perp$ is a fixed object, chosen to become dualizing. A morphism from $\langle A, B, \phi \rangle$ to $\langle C, D, \gamma \rangle$ was defined as a pair $\langle u : A \to C, B \leftarrow D : v \rangle$ of $\mathcal{V}$-arrows, making the square

$$\begin{tikzcd} A \otimes D \arrow[r, "A\otimes v"] \arrow[d, "u\otimes D"'] & A \otimes B \arrow[d, "\phi"] \\ C \otimes D \arrow[r, "\gamma"'] & \perp \end{tikzcd} \tag{1}$$

commute. This is the setting in which the autonomous structure of a Chu category was originally discovered.

The starting point of the present paper is the fact that the category described by Chu is isomorphic to the comma category $\mathcal{V}/\perp\!\!\!\perp$, induced by the homming functor

$$\perp\!\!\!\perp : \mathcal{V}^{op} \longrightarrow \mathcal{V} \quad : \quad A \longmapsto A^{\perp} = A \multimap \perp \, . \tag{2}$$

By definition, the objects of $\mathcal{V}/\perp\!\!\!\perp$ (i.e. $\mathrm{Id}_{\mathcal{V}}/\perp\!\!\!\perp$, sometimes written $\mathrm{Id}_{\mathcal{V}} \downarrow \perp\!\!\!\perp$ (Mac Lane 1971, II.6)) are the triples $\langle A, A \xrightarrow{f} B^{\perp}, B\rangle$. Clearly, each such triple corresponds by transposition to unique Chu's triple $\langle A, B, A \otimes B \xrightarrow{\phi} \perp\rangle$. Furthermore, a $\mathcal{V}/\perp\!\!\!\perp$-morphism from $\langle A, f, B\rangle$ to $\langle C, g, D\rangle$ is a pair $\langle u : A \to C, B \leftarrow D : v\rangle$, making the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B^{\perp} \arrow[d, "v^{\perp}"] \\ C \arrow[r, "g"'] & D^{\perp} \end{tikzcd} \tag{3}$$

commute. But (1) commutes if and only if (3) commutes. The morphisms thus coincide, and the categories are isomorphic. Presenting the Chu construction on the background of the comma construction opens up the road to its couniversal property.

The focus of the present paper is thus the couniversality of the comma construction. While it was formally introduced only in the early sixties, at the beginning of Lawvere's thesis (Lawvere 1963), the germ of the comma construction can actually be found already in the treatment of dual pairs in Mackey's 1942 thesis (Mackey 1945) — which is the remote source of the Chu construction too.

Finally, let us mention that the Dialectica categories (de Paiva 1989; Brown and Gurr 1991) can also be reduced to the comma construction, but poset-enriched. And this enriched version can be derived from arrow 2-categories, which will be descussed in the following section.

2. Comma comonad

Let $\mathcal{C}$ be any category and $\overrightarrow{\mathcal{C}} = \mathcal{C}/\mathcal{C}$ the category of its arrows. The embedding $\mathsf{I} : \mathcal{C} \to \overrightarrow{\mathcal{C}}$, taking each object of $\mathcal{C}$ to the identity on it, has both adjoints: the domain and the codomain functors. Hence a monad and a comonad on $\overrightarrow{\mathcal{C}}$, both idempotent. It is easy to see that the algebra, as well as the coalgebra structure on an arrow $v \in \overrightarrow{\mathcal{C}}$ boils down to the inverse arrow v^{-1}. The category of algebras, as well as the category of coalgebras, is isomorphic to the subcategory of $\overrightarrow{\mathcal{C}}$ spanned by the isomorphisms. But this is the essential image of the embedding $\mathsf{I} : \mathcal{C} \to \overrightarrow{\mathcal{C}}$ — which is thus monadic and comonadic.

The story gets more interesting when $\mathcal{C}$ is a 2-category. The arrow 2-category $\overrightarrow{\mathcal{C}}$ has, of course, the arrows $V : \mathcal{V}' \to \mathcal{V}''$, $U : \mathcal{U}' \to \mathcal{U}''$ of $\mathcal{C}$ as objects. A $\overrightarrow{\mathcal{C}}$-arrow from V to

U will now be a square with a 2-cell

$$
\begin{tikzcd}
\mathcal{V}' \arrow[r, "F'"] \arrow[d, "V"'] & \mathcal{U}' \arrow[d, "U"] \arrow[dl, phantom, "\swarrow\varphi"] \\
\mathcal{V}'' \arrow[r, "F'"'] & \mathcal{U}''
\end{tikzcd}
\tag{4}
$$

— that is, a triple $F = \langle F', \varphi, F'' \rangle$. Its composite with a morphism $E = \langle E', \epsilon, E'' \rangle$ from U to T will be

$$
E \cdot F \quad = \quad \langle E'F',\ E''\varphi \circ \epsilon F',\ E''F'' \rangle \tag{5}
$$

On the other hand, a 2-cell $\lambda : F \to G$ in $\overrightarrow{\mathcal{C}}$, where $G = \langle G', \gamma, G'' \rangle$ is another morphism from V to U, will be a pair $\langle \lambda', \lambda'' \rangle$ of 2-cells from $\mathcal{C}$, preserving φ and γ, in the sense that the following square commutes.

$$
\begin{tikzcd}
UF' \arrow[r, "U\lambda'"] \arrow[d, "\varphi"'] & UG' \arrow[d, "\gamma"] \\
F''V \arrow[r, "\lambda''V"'] & G''V
\end{tikzcd}
\tag{6}
$$

The composition of 2-cells of $\overrightarrow{\mathcal{C}}$ is componentwise, i.e. inherited from $\mathcal{C}$.

The described embedding $\mathsf{I} : \mathcal{C} \to \overrightarrow{\mathcal{C}}$ obviously extends to these 2-cells. How about its adjoints? To avoid unnecessary abstraction, let us restrict to the 2-category $\mathcal{C} = \mathsf{CAT}$ of categories, functors and natural transformations. The right adjoint $\mathsf{D} : \overrightarrow{\mathsf{CAT}} \to \mathsf{CAT}$ of I is now induced by the one-sided comma construction. This is the "glorified" domain functor. The left adjoint, corresponding to the codomain functor, is less familiar, but can be spelled out as an exercise.

Anyway, the right adjoint D takes each functor $V : \mathcal{V}' \to \mathcal{V}''$ to the comma category $\mathsf{D}V = V/\mathcal{V}''$. The objects of $\mathsf{D}V$ are thus the triples $\langle X', x, X'' \rangle$, while the arrows are pairs $\langle u', u'' \rangle$, which make the square below commutative.

$$
\begin{tikzcd}
\langle X', \arrow[d, "u'"'] & VX' \arrow[r, "x"] \arrow[d, "Vu'"'] & X'', \arrow[d, "u''"] & X'' \rangle \arrow[d, "u''"] \\
\langle Y', & VY' \arrow[r, "y"'] & Y'', & Y'' \rangle
\end{tikzcd}
\tag{7}
$$

Note that $V/\mathcal{V}''$ is isomorphic to $\mathcal{V}'$ if and only if $\mathcal{V}''$ is a discrete category, i.e. set. Restricted to sets and functions, D thus boils down to the domain functor.

Given a $\overrightarrow{\mathsf{CAT}}$-morphism $F : V \to U$, the construction D induces a functor $\mathsf{D}F : \mathsf{D}V \to \mathsf{D}U$, which takes each object $\langle X', VX' \stackrel{x}{\to} X'', X'' \rangle$ of $V/\mathcal{V}''$ to

$$
\langle F'X',\ \ UF'X' \stackrel{\varphi}{\longrightarrow} F''VX' \stackrel{F''x}{\longrightarrow} F''X'',\ \ F''X'' \rangle, \tag{8}
$$

in $U/\mathcal{U}''$, while an arrow $\langle u', u'' \rangle$ goes to $\langle F'u', F''u'' \rangle$. Finally, the image $\mathsf{D}\lambda : \mathsf{D}F \to \mathsf{D}G$ of a 2-cell $\lambda : F \to G$ in $\overrightarrow{\mathsf{CAT}}$ is obtained simply by "changing the perspective": condition (6) ensures that the pair $\langle \lambda', \lambda'' \rangle$ forms a natural family of $U/\mathcal{U}''$-morphisms, which can be construed as a transformation $\mathsf{D}F \to \mathsf{D}G : \mathsf{D}V \to \mathsf{D}U$.

The couniversal property of the comma construction now induces an equivalence (even isomorphism) of hom-categories

$$\overrightarrow{\mathsf{CAT}}\,(\mathsf{I}\mathcal{A}, V) \quad \simeq \quad \mathsf{CAT}\,(\mathcal{A}, \mathsf{D}V)\,, \tag{9}$$

natural in $\mathcal{A}$ and V. This is the adjunction $\mathsf{I} \dashv D$. Hence the comonad $\mathsf{P} = \mathsf{I} \cdot \mathsf{D}$ on $\overrightarrow{\mathsf{CAT}}$, with the data

$$V \xleftarrow{\ \mathsf{e}_V\ } \mathsf{P}V \xrightarrow{\ \mathsf{d}_V\ } \mathsf{P}^2 V$$

$$\begin{tikzcd}
\mathcal{V}' \arrow[d, "V"'] \arrow[dr, phantom, "\epsilon"] & V/\mathcal{V}'' \arrow[l, "E'"'] \arrow[r, "\mathsf{I}"] \arrow[d, "\mathrm{Id}"] \arrow[dr, phantom, "\mathrm{id}"] & \overline{V/\mathcal{V}''} \arrow[d, "\mathrm{Id}"] \\
\mathcal{V}'' & V/\mathcal{V}'' \arrow[l, "E''"] \arrow[r, "\mathsf{I}"'] & \overline{V/\mathcal{V}''}
\end{tikzcd} \tag{10}$$

$$\begin{aligned}
E'\langle X', x, X'' \rangle &= X' \\
\epsilon\langle X', x, X'' \rangle &= x : VX' \to X'' \\
E''\langle X', x, X'' \rangle &= X''.
\end{aligned}$$

It turns out that a functor $V : \mathcal{V}' \to \mathcal{V}''$ has a P-coalgebra structure if and only if it is an equivalence. But equivalences span the essential image of I, just like isomorphisms span it for ordinary categories.

Proposition 2.1. The embedding $\mathsf{I} : \mathsf{CAT} \to \overrightarrow{\mathsf{CAT}}$ is comonadic.

As always, this means that the comparison functor Υ from CAT to the 2-category $\overrightarrow{\mathsf{CAT}}_{\mathsf{P}}$ of P-coalgebras is an equivalence. But note that all the notions involved: functor, coalgebra, equivalence — need to be suitably enriched† for this 2-dimensional setting. In the presence of nontrivial 2-cells, the defining diagrams for coalgebras and their homomorphisms (Mac Lane 1971, VI.1(1^{op})) can be required to commute strictly, or up to coherent isomorphisms, or up to arbitrary coherent 2-cells. Hence strict, pseudo and lax coalgebras. Proposition 2.1 should be taken in the pseudo sense, i.e. up to coherent isomorphisms. Although P is a comonad in the strictest possible sense, its strict coalgebras are not well-behaved: e.g., they can be naturally isomorphic, but have no strict coalgebra homomorphisms between them.

A P-coalgebra $K : V \to \mathsf{P}V$ will thus be given together with a pair of invertible 2-cells

† As in enriched category theory (Kelly 1982), the ordinary terminology is simply lifted to higher dimensions: the most fundamental notions usually have unique liftings.

in $\overrightarrow{\mathsf{CAT}}$

$$\begin{tikzcd} V \arrow[r, "K"] \arrow[d, "K"'] \arrow[dr, phantom, "\overset{\theta}{\cong}"] & \mathsf{P}V \arrow[d, "\mathsf{d}_V"] \\ \mathsf{P}V \arrow[r, "\mathsf{P}K"'] & \mathsf{P}^2V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[d, "K"'] \arrow[dr, "\mathrm{Id}"] \arrow[d, phantom, shift left=4ex, "\overset{\zeta}{\cong}"] & \\ \mathsf{P}V \arrow[r, "\mathsf{e}_V"'] & V \end{tikzcd} \tag{11}$$

They are required to be coherent, in the sense that all diagrams generated by them commute. A set of conditions which ensure this can be found in (Street 1974; Zöberlein 1976). The filler θ realizes the so called *chain* condition, and ζ is the *counit* condition.

On the other hand, if $L : U \to \mathsf{P}U$ is another coalgebra, a homomorphism $\Phi : K \to L$ will be a $\overrightarrow{\mathsf{CAT}}$-morphism $F : V \to U$, accompanied with an invertible 2-cell

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\overset{\phi}{\cong}"] & \mathcal{U} \arrow[d, "L"] \\ \mathsf{P}V \arrow[r, "\mathsf{P}F"'] & \mathsf{P}\mathcal{U} \end{tikzcd} \tag{12}$$

required to be coherent with the companions θ, ζ of K and L. With the 2-cells inherited from $\overrightarrow{\mathsf{CAT}}$, this time coherent with companions (12), P-coalgebras form the 2-category $\overrightarrow{\mathsf{CAT}}_{\mathsf{P}}$.

We shall now show that the canonical comparison functor $\Upsilon : \mathsf{CAT} \to \overrightarrow{\mathsf{CAT}}_{\mathsf{P}}$ is an equivalence, i.e. fully faithful and essentially surjective. The former means that there is a natural equivalence

$$\mathsf{CAT}\,(\mathcal{A}, \mathcal{B}) \quad \simeq \quad \overrightarrow{\mathsf{CAT}}_{\mathsf{P}}\,(\Upsilon\mathcal{A}, \Upsilon\mathcal{B})\,. \tag{13}$$

The essential surjectivity means that each P-coalgebra is equivalent to one in the form $\Upsilon\mathcal{A}$, for some category $\mathcal{A}$.

Proof of proposition 2.1. Clearly, Υ must take each category $\mathcal{A}$ to the identity on it, with the structure map

$$\mathrm{Id} \xrightarrow{\Upsilon\mathcal{A}} \mathsf{P}\,(\mathrm{Id})$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathsf{I}"] \arrow[d, "\mathrm{Id}"'] \arrow[dr, phantom, "\mathrm{id}"] & \overrightarrow{\mathcal{A}} \arrow[d, "\mathrm{Id}"] \\ \mathcal{A} \arrow[r, "\mathsf{I}"'] & \overrightarrow{\mathcal{A}} \end{tikzcd} \tag{14}$$

Check that this is indeed a P-coalgebra, and that (13) will hold, is straightforward. So Υ is full and faithful. Towards a proof that it is essentially surjective, we need to show that the underlying functor $V : \mathcal{V}' \to \mathcal{V}''$ of a P-coalgebra $K : V \to \mathsf{P}V$ is always an equivalence. This will yield the $\overrightarrow{\mathsf{CAT}}$-equivalence $V \simeq \mathrm{Id}_{\mathcal{V}''}$

$$
\begin{tikzcd}
\mathcal{V}' \arrow[r, "V"] \arrow[d, "V"'] & \mathcal{V}'' \arrow[d, "\mathrm{Id}"] \arrow[dl, phantom, "\swarrow \mathrm{id}"] \\
\mathcal{V}'' \arrow[r, "\mathrm{Id}"'] & \mathcal{V}''
\end{tikzcd}
\tag{15}
$$

— which is a coalgebra homomorphism. Hence the required $\overrightarrow{\mathsf{CAT}}_{\mathsf{P}}$-equivalence between K and $\Upsilon\mathcal{V}''$.

So let $K : V \to \mathsf{P}V$ be an arbitrary coalgebra. As a $\overrightarrow{\mathsf{CAT}}$-morphism, K is a triple $\langle K', \kappa, K''\rangle$ of two functors and a natural transformation, as on the next diagram. Their further decomposition, displayed below, is induced by the couniversal property of the comma construction.

$$
\begin{tikzcd}
\mathcal{V}' \arrow[r, "K'"] \arrow[d, "V"'] & V/\mathcal{V}'' \arrow[d, "\mathrm{Id}"] \arrow[dl, phantom, "\swarrow \kappa"] \\
\mathcal{V}'' \arrow[r, "K''"'] & V/\mathcal{V}''
\end{tikzcd}
\tag{16}
$$

$$
\begin{array}{rcl}
K' & = & \langle K_0' : \mathcal{V}' \to \mathcal{V}', \quad VK_0' \stackrel{k'}{\to} K_1', \quad K_1' : \mathcal{V}' \to \mathcal{V}''\rangle \\
\kappa & = & \langle \kappa_0 : K_0' \to K_0''V, \quad \kappa_1 : K_1' \to K_1''V\rangle \\
K'' & = & \langle K_0'' : \mathcal{V}'' \to \mathcal{V}', \quad VK_0'' \stackrel{k''}{\to} K_1'', \quad K_1'' : \mathcal{V}'' \to \mathcal{V}''\rangle
\end{array}
$$

The commutativity condition on the $V/\mathcal{V}''$-morphisms which form κ tells that

$$
\begin{tikzcd}
VK_0' \arrow[r, "k'"] \arrow[d, "V\kappa_0"'] & K_1' \arrow[d, "\kappa_1"] \\
VK_0''V \arrow[r, "k''V"'] & K_1''V
\end{tikzcd}
\tag{17}
$$

must commute. Note that the natural transformation $VK_0' \to K_1''V$, shown on this diagram, also appears as the companion $E''\kappa \circ \epsilon K' = \kappa_1 \circ k'$ of the $\overrightarrow{\mathsf{CAT}}$-morphism $\mathsf{e} \cdot K$, from the counit condition.

On top of all this, the coalgebra structure $K : V \to \mathsf{P}V$ includes the invertible 2-cells θ and ζ, supporting the chain and the counit conditions (11). In a sense, they are the most important part here, since they actually force V to be an equivalence.

Since the morphism $\mathsf{e}\cdot K$ boils down to $\langle K_0', \kappa_1 \circ k', K_1''\rangle$, the invertible 2-cell $\zeta : \mathrm{Id} \to \mathsf{e}\cdot K$ yields the natural isomorphisms $\zeta' : \mathrm{Id}_{\mathcal{V}'} \to K_0'$ and $\zeta'' : \mathrm{Id}_{\mathcal{V}''} \to K_1''$. They must satisfy condition (6), i.e.

$$\kappa_1 \circ k' \circ V\zeta' \quad = \quad \zeta'' V. \tag{18}$$

The transformation $k''V \circ V\kappa_0 = \kappa_1 \circ k'$ from (17) is thus an isomorphism. It can actually be made into an identity by precomposing with $V\zeta'$ and postcomposing with the inverse of $\zeta''V$. Without loss of generality, we can thus assume that K_0' and K_1'' are both identities — on $\mathcal{V}'$ and $\mathcal{V}''$ respectively. If they are not, transfer them along ζ', resp. ζ''. Of course, this changes the transformations k', k'', κ_0 and κ_1 as well, and the composite transformation on (17) becomes identity on V.

The 2-cell θ, on the other hand, decomposes into natural isomorphisms $\theta' : \mathsf{D}K\cdot K' \longrightarrow \mathsf{I}\cdot K'$ and $\theta'' : \mathsf{D}K\cdot K'' \longrightarrow \mathsf{I}\cdot K''$. Each of them is a family of morphisms in $\overrightarrow{V/\mathcal{V}''}$. This is a comma category over a comma category: the components of θ' and θ'' are pairs of $V/\mathcal{V}''$-arrows — i.e. *quadruples* of $\mathcal{V}''$-arrows. We will not write them all out, but just outline the way in which they lead to the conclusion that we are seeking.

Firstly, the domain part of θ'' makes the arrow

$$K_0''k'' \circ \kappa_0 K_0'' \quad : \quad K_0'' \to K_0''VK_0'' \to K_0'' \tag{19}$$

(as an object of an arrow category) isomorphic to the identity. In other words, this arrow must be an isomorphism. Together with the equation $k''V \circ V\kappa_0 = \mathrm{id}_V$, derived from (18), this implies that K_0'' is right adjoint to V, with the unit $\kappa_0 : \mathrm{Id} \to K_0''V$ and the counit $k'' \circ Vi : VK_0'' \to \mathrm{Id}$, where $i : K_0'' \to K_0''$ is the inverse of isomorphism (19). Cf. (Pavlović 1995, appendix C).

In a similar fashion, the codomain part of θ' ensures that the transformation $k' \circ \kappa_1 : K_1' \to V \to K_1'$ is an isomorphism. As $\kappa_1 \circ k' = \mathrm{id}_{\mathrm{V}}$ has already been derived from (18), k' and κ_1 turn out to be each other's inverses. We can now transfer K_1' along $k' : V \to K_1'$ and make it equal with V. Without loss of generality, the transformations k' and κ_1 can be taken to be identities.

With the structure of K simplified like this, it is not hard to see that the domain part of θ' forces $\kappa_0 : \mathrm{Id} \to K_0''V$ to be an isomorphism, while the codomain part of θ'' does the same with $k'' : VK_0'' \to \mathrm{Id}$. Since these transformations are essentially the unit and the counit of the adjunction $V \dashv K_0''$, we conclude that V is an equivalence. $\square$

Is P *a* KZ-*comonad?* The preceding analysis of coalgebra $K : V \to \mathsf{P}V$ shows that it is completely determined, up to isomorphism, by the underlying functor V. Namely, V carries a P-coalgebra structure, essentially unique, if and only if it is an equivalence. Being a P-coalgebra is thus a *property* of V, rather than added structure. The category of P-coalgebras in $\overrightarrow{\mathsf{CAT}}$ is thus equivalent with a subcategory of $\overrightarrow{\mathsf{CAT}}$, and we have a comonadic *embedding*, rather than just a comonadic forgetful functor. Indeed, the functor I does not "forget" any structure, but actually *localises* $\mathcal{C}$ in $\overrightarrow{\mathcal{C}}$.

In ordinary category theory, situations like this are captured by *idempotent* monads and comonads (Appelgate and Tierney 1969). They respectively localise reflective and coreflective subcategories in a given category, i.e. extract a property rather than impose a structure. In 2-category theory, though, the (co)monads property extractiong (co)monads

— those that allow essentially unique (co)algebra structures — may not be idempotent in the ordinary sense. For instance, various free completions of categories significantly enlarge any given input; yet they extract as algebras just the suitably complete categories. Such constructions typically form *KZ*-monads. The name has been coined by Street (Street 1974; Street 1980), from Kock's (Kock 1972–95) and Zöberlein's (Zöberlein 1976) initials. A different name has been used in (Blackwell *et al.* 1989, 6.5).

While an ordinary idempotent comonad G forces each structure map $K : V \to \mathsf{G}V$ to be the inverse of the counit $\mathsf{e} : \mathsf{G}V \to V$, in the 2-dimensional theory, a structure map for a KZ-comonad G must be right adjoint to the counit. Abstractly, a KZ-comonad is recognized by the presence of a 2-cell

$$\rho : \mathsf{eG} \to \mathsf{Ge} \tag{20}$$

which yields identities whenever composed with either the comultiplication d or the counit e of G[‡]. In an ordinary category, such a 2-cell degenerates into the equality $\mathsf{eG} = \mathsf{Ge}$, which a characteristic of the idempotent comonads. So it seems that the KZ-(co)monads are *the* generalisation of the idempotent ones in the 2-dimensional setting and that any property-extracting (co)monad on a 2-category should be KZ. What else could enforce the essential uniqueness of the structure maps if not adjunction?

Well, the comma comonad $\mathsf{P} : \overrightarrow{\mathsf{CAT}} \to \overrightarrow{\mathsf{CAT}}$ provides an answer. P is *not* KZ, although it certainly extends an idempotent comonad, namely the one derived from the domain functor, as explained in the beginning. As demonstrated above, a P-coalgebra $K : V \to \mathsf{P}V$ is essentially unique and denotes a property of V. But K is generally not adjoint to the counit $\mathsf{e}_V : \mathsf{P}V \to V$.

To show this, consider the case when V is the identity on $\mathcal{V}$. The functor $\mathsf{P}V$ is then the identity on the arrow category $\overrightarrow{\mathcal{V}}$. The coalgebra structure $K : \mathrm{Id} \to \mathsf{P}(\mathrm{Id})$ is $\langle \mathsf{I}, \mathrm{id}, \mathsf{I}\rangle$, where $\mathsf{I} : \mathcal{V} \to \overrightarrow{\mathcal{V}}$, as before, takes each object to its identity. The counit $\mathsf{e}_{\mathrm{Id}} : \mathsf{P}(\mathrm{Id}) \to \mathrm{Id}$, on the other hand, consists of the domain and the codomain functors, with the obvious natural transformation between them, as on (10). A direct calculation now shows that, for instance, for $\mathcal{V} = \mathsf{Set}$, there are no 2-cells $K \cdot \mathsf{e} \to \mathrm{Id}$, or $\mathrm{Id} \to K \cdot \mathsf{e}$. In fact, K and e are adjoint if and only if $\mathcal{V}$ is a groupoid.

Remark. For a general 2-category $\mathcal{C}$, the right adjoint D of $\mathsf{I} : \mathcal{C} \to \overrightarrow{\mathcal{C}}$ exists if and only if $\mathcal{C}$ is representable, in the sense of (Street 1974). With this assumption, proposition 2.1 goes through unchanged.

3. Duality comonad

What does all this have to do with the Chu construction?

The idea of the Chu construction is to transform an autonomous category $\mathcal{V}$ into a

[‡] A KZ-comonad is thus a quadruple $(\mathsf{G}, \mathsf{d}, \mathsf{e}, \rho)$. In fact, had we chosen to call comonads *cotriples* (the terminology which probably prevails in the literature), we would now be able to call KZ-comonads — *coquadruples*. On the other hand, had we chosen to distinguish 2-dimensional monads as *doctrines*, we would now have KZ-codoctrines, and could say in 2.1 that I is *codoctrinary*, or even *codoctrinable*.

$*$-autonomous, so that a chosen object $\perp$ becomes dualising. In other words, the internal homming into $\perp$ is to be developed into an equivalence.

On the other hand, the comma comonad, as we have just seen, transforms arbitrary functors into equivalences. It is thus not a coincidence that every Chu category is isomorphic to a comma category. And the couniversal property of the Chu construction is a consequence of the couniversal property of the comma construction. But there are some subtle points.

3.1. *Self-adjunctions*

Roughly, the idea is to get the Chu comonad by restricting the comma comonad P to the subcategory of $\overrightarrow{\mathsf{CAT}}$ spanned by the homming endofunctors on autonomous categories. But we shall take one step at a time, and first inquire how to restrict to contravariant endofunctors $V : \mathcal{V} \to \mathcal{V}^{op}$ in general.

Formally, this restriction can be viewed as equalizing the functors Dom and Cod^{op} from $\overrightarrow{\mathsf{CAT}}$ to CAT, which map $V : \mathcal{V}' \to \mathcal{V}''$ to $\mathcal{V}'$ and to $\mathcal{V}''^{op}$ respectively. Of course, they send a $\overrightarrow{\mathsf{CAT}}$-morphism $\langle F', F'', \varphi\rangle : V \to U$ to $F' : \mathcal{V}' \to \mathcal{U}'$ and $F''^{op} : \mathcal{V}''^{op} \to \mathcal{U}''^{op}$ respectively; and a 2-cell $\langle \lambda', \lambda''\rangle : F \to G$ is projected to $\lambda' : F' \to G'$ by Dom and to $\lambda''^{op} : G''^{op} \to F''^{op}$ by Cod^{op}. The latter functor thus reverts the direction of the 2-cells, and the former does not. In order to be able equalize them, we must begin by restricting $\overrightarrow{\mathsf{CAT}}$ to invertible 2-cells.

Having done all this, we end up with the subcategory of $\overrightarrow{\mathsf{CAT}}$ consisting of the objects in the form $V : \mathcal{V} \to \mathcal{V}^{op}$, the morphisms in the form $\langle F, F^{op}, \varphi\rangle$, and the 2-cells $\langle \lambda, (\lambda^{-1})^{op}\rangle$, where $\lambda : F \to G$ is a natural isomorphism and $\lambda^{-1} : G \to F$ its inverse, yielding $(\lambda^{-1})^{op} = (\lambda^{op})^{-1} : F^{op} \to G^{op}$. The only trouble is that the comma comonad P, as described in (10), cannot be directly restricted to this subcategory of *contravariant* endofunctors, since the image of P consists of the *identity* functors. Therefore, P must be slightly modified.

To see how, observe that applying the comma construction to an endofunctor $V : \mathcal{V} \to \mathcal{V}^{op}$ yields a contravariant equivalence, *duality*

$$V/\mathcal{V}^{op} \quad \tilde{\rightarrow} \quad (V/\mathcal{V}^{op})^{op} \tag{21}$$

— as soon as V is *self-adjoint.* Namely, a self-adjunction $V \dashv V^{op}$ can be characterized by the existence of an equivalence

$$V/\mathcal{V}^{op} \quad \tilde{\rightarrow} \quad \mathcal{V}/V^{op}, \tag{22}$$

commuting with the projections to $\mathcal{V} \times \mathcal{V}^{op}$. (With an isomorphism instead of the equivalence, this characterisation was already in (Lawvere 1963) — where the comma construction was actually introduced. The fact that an equivalence will do follows from (Pavlović 1995, lemma C.1).) Since $\mathcal{V}/V^{op}$ is clearly isomorphic with $(V/\mathcal{V}^{op})^{op}$, (22) yields (21). Conversely, a duality (21), commuting with the projections to $\mathcal{V} \times \mathcal{V}^{op}$, can exist only if $V \dashv V^{op}$.

Therefore, $\overrightarrow{\mathsf{CAT}}$ must be restricted to the category SA of *self-adjunctions.* To minimize the accumulation of the *"op"* superscripts, we shall consider them in the form V^{op} :

$\mathcal{V}^{op} \to \mathcal{V}$ rather than $V : \mathcal{V} \to \mathcal{V}^{op}$; and we shall generically write $\perp\!\!\!\perp$ instead of V^{op}. A self-adjunction will thus consist of a category $\mathcal{V}$, equipped with a functor $\perp\!\!\!\perp : \mathcal{V}^{op} \to \mathcal{V}$ and a natural family $\eta A : A \to A^{\perp\perp}$ satisfying

$$(\eta A)^{\perp} \circ \eta(A^{\perp}) \quad = \quad \mathrm{id}_A \tag{23}$$

where $X^{\perp}$ denotes $\perp\!\!\!\perp X$, viewed as well and object of $\mathcal{V}$, as well as the same object $\perp\!\!\!\perp^{op} X$ in $\mathcal{V}^{op}$. It is easy to show that η induces the natural bijection

$$\mathcal{V}(A, B^{\perp}) \quad \cong \quad \mathcal{V}(B, A^{\perp}). \tag{24}$$

The SA-morphisms are extracted from $\overrightarrow{\mathsf{CAT}}$ by equalizing the domain and the codomain components, as outlined above. Given another category $\mathcal{U}$, again with $\perp\!\!\!\perp : \mathcal{U}^{op} \to \mathcal{U}$ and $\eta B : B \to B^{\perp\perp}$, an SA-morphism will thus be a functor $F : \mathcal{V} \to \mathcal{U}$, accompanied with a natural family $\varphi A : F(A^{\perp}) \to (FA)^{\perp}$, preserving the adjunction, i.e. making the diagram

$$\begin{tikzcd} FA \arrow[r, "F\eta A"] \arrow[d, "\eta FA"'] & F(A^{\perp\perp}) \arrow[d, "\varphi(A^{\perp})"] \\ (FA)^{\perp\perp} \arrow[r, "(\varphi A)^{\perp}"'] & (F(A^{\perp}))^{\perp} \end{tikzcd} \tag{25}$$

commute. If $E : \mathcal{U} \to \mathcal{W}$ is another such morphism, accompanied with $\epsilon B : E(B^{\perp}) \to (EB)^{\perp}$, the composite $E \cdot F$ will be accompanied with

$$EF(A^{\perp}) \stackrel{E\varphi A}{\longrightarrow} E(FA)^{\perp} \stackrel{\epsilon FA}{\longrightarrow} (EFA)^{\perp}, \tag{26}$$

just as (5) suggests.

Finally, given SA-morphisms $F, G : \mathcal{V} \to \mathcal{U}$ a 2-cell between them will be a natural isomorphism $\lambda : F \stackrel{\sim}{\to} G$, coherent with the companions φ and γ, in the sense that

$$\begin{tikzcd} F(A^{\perp}) \arrow[r, "\lambda(A^{\perp})"] \arrow[d, "\varphi A"'] & G(A^{\perp}) \arrow[d, "\gamma A"] \\ (FA)^{\perp} & (GA)^{\perp} \arrow[l, "(\lambda A)^{\perp}"] \end{tikzcd} \tag{27}$$

commutes. This condition is, of course, derived from (6).

The objects and the arrows of SA will usually be denoted by the names of their underlying categories $\mathcal{V}$, $\mathcal{U}$, and by the names of the underlying functors F, G.

3.2. *Dualities*

A *duality*, is a self-adjunction which happens to be an equivalence. In fact, a functor $* : \mathcal{D}^{op} \to \mathcal{D}$ is duality if and only if there is an isomorphism

$$A \quad \cong \quad A^{**} \tag{28}$$

natural in A. The existence of such an isomorphism implies that $*$ has to be self-adjoint (Pavlović 1995, lemma C.1 again) and that its unit $\eta A : A \to A^{**}$ must be an isomorphism too (Johnstone and Moerdijk 1989, lemma 1.3.). The duality functors will invariably be denoted by $*$.

The morphisms of dualities will be the functors that preserve them. In other words, a duality morphism $G : \mathcal{D} \to \mathcal{E}$ should come accompanied with a natural iso

$$\gamma A \quad : \quad G(A^*) \stackrel{\sim}{\to} (GA)^*, \tag{29}$$

coherent with the units $\eta A : A \stackrel{\sim}{\to} A^{**}$ of the equivalences $*$ on $\mathcal{D}$ and $\mathcal{E}$. This coherence condition is a special case of (25), and a duality morphism is just a morphism of self-adjunctions which happens to be accompanied by an iso. Dualities and their morphisms form a subcategory DU of SA, full on the 2-cells, but not on the 1-cells.

3.3. *The comonad*

Now we want to show that the inclusion $\mathsf{U} : \mathsf{DU} \hookrightarrow \mathsf{SA}$ is comonadic. We first construct its right adjoint $\mathsf{C} : \mathsf{SA} \to \mathsf{DU}$, and then show that the category of coalgebras for the induced comonad $\mathsf{G} = \mathsf{U} \cdot \mathsf{C}$ is equivalent with DU.

The object part of C is based on (21). Given a self-adjunction $\mathcal{V}$, the underlying category of the duality $\mathsf{C}\mathcal{V}$ will be the dual comma

$$\mathsf{C}\mathcal{V} \quad = \quad (\mathcal{V}/\perp)^{op}. \tag{30}$$

The duality functor $* : (\mathsf{C}\mathcal{V})^{op} \to \mathsf{C}\mathcal{V}$ is now induced by the transposition along (24)

$$\langle A, A \stackrel{f}{\to} B^\perp, B\rangle^* \quad = \quad \langle B, B \stackrel{f^*}{\to} A^\perp, A\rangle, \tag{31}$$

— where $f^* : B \to A^\perp$ denotes the transpose of $f : A \to B^\perp$. On morphisms $\langle u, v\rangle : \langle A, f, B\rangle \to \langle C, g, D\rangle$, this duality just switches the components:

$$\begin{tikzcd} \langle A, & A \arrow[r, "f"] & B^\perp, & B\rangle \arrow[d, "v"] \\ \langle C, \arrow[u, "u"'] & C \arrow[u, "u"'] \arrow[r, "g"'] & D^\perp, \arrow[u, "v^\perp"] & D\rangle \end{tikzcd} \tag{32}$$

$$\langle u, v\rangle^* \quad = \quad \langle v, u\rangle : \langle D, g^*, C\rangle \to \langle B, f^*, A\rangle. \tag{33}$$

Note that the unit $\eta A : A \to A^{**}$ is indentity, which means that the functor $*$ is actually an isomorphism.

The arrow part of C, on the other hand, is based on the arrow part of the functor

D from section 2. An SA-morphism $F : \mathcal{V} \to \mathcal{U}$ of self-adjunctions, will thus induce a functor $\mathsf{C}F : \mathsf{C}\mathcal{V} \to \mathsf{C}\mathcal{U}$ defined as follows:

$$\mathsf{C}F\langle A, A \xrightarrow{f} B^\perp, B\rangle \;=\; \langle FA,\; FA \xrightarrow{Ff} F(B^\perp) \xrightarrow{\varphi} (FB)^\perp, FB\rangle \tag{34}$$

$$\mathsf{C}F\langle u, v\rangle \;=\; \langle Fu, Fv\rangle \tag{35}$$

Condition (25) ensures that this is a DU-morphism. In the first instance, it says that

$$\begin{tikzcd}
FB \arrow[r, "F\eta B"] \arrow[d, "\eta FB"'] & F(B^{\perp\perp}) \arrow[r, "F(f^\perp)"] \arrow[d, "\varphi(B^\perp)"] & F(A^\perp) \arrow[d, "\varphi A"] \\
(FB)^{\perp\perp} \arrow[r, "(\varphi B)^\perp"'] & (F(B^\perp))^\perp \arrow[r, "(Ff)^\perp"'] & (FA)^\perp
\end{tikzcd} \tag{36}$$

commutes for every $f : A \to B^\perp$. Via standard transpositions

$$\begin{aligned} f^\perp \circ \eta \;&=\; f^* \text{ and} \\ (\varphi B \circ Ff)^\perp \circ \eta FB \;&=\; (\varphi B \circ Ff)^*, \end{aligned}$$

diagram (36) yields

$$\varphi A \circ F(f^*) \;=\; (\varphi B \circ Ff)^*. \tag{37}$$

But this equation implies that $\mathsf{C}F$ preserves the duality — on the nose:

$$\begin{aligned}
\mathsf{C}F\langle A, A \xrightarrow{f} B^\perp, B\rangle^* \;&\overset{(31)}{=}\; \mathsf{C}F\langle B, B \xrightarrow{f^*} A^\perp, A\rangle \\
&\overset{(34)}{=}\; \langle FB, FB \xrightarrow{Ff^*} F(A^\perp) \xrightarrow{\varphi} (FA)^\perp, FA\rangle \\
&\overset{(37)}{=}\; \langle FB, FB \xrightarrow{(\varphi\circ Ff)^*} (FA)^\perp, FA\rangle \\
&\overset{(31)}{=}\; \langle FA, FA \xrightarrow{Ff} F(B^\perp) \xrightarrow{\varphi} (FB)^\perp, FB\rangle^* \\
&\overset{(34)}{=}\; \Big(\mathsf{C}F\langle A, A \xrightarrow{f} B^\perp, B\rangle\Big)^*.
\end{aligned} \tag{38}$$

With identity as its companion, $\mathsf{C}F$ is thus a duality morphism.

A 2-cell $\lambda : F \to G$ from SA induces in DU a 2-cell $\mathsf{C}\lambda : \mathsf{C}F \to \mathsf{C}G$, with the components $(\mathsf{C}\lambda)\langle A, f, B\rangle = \langle(\lambda A)^{-1}, \lambda B\rangle$.

$$\begin{tikzcd}
\langle FA, & FA \arrow[r, "Ff"] & FB^\perp \arrow[r, "\varphi"] \arrow[d, "\lambda(B^\perp)"] & (FB)^\perp, & FB\rangle \arrow[d, "\lambda B"] \\
\langle GA \arrow[u, "(\lambda A)^{-1}"] & GA \arrow[r, "Gf"'] \arrow[u, "(\lambda A)^{-1}"] & GB^\perp \arrow[r, "\varphi'"'] & (GB)^\perp, \arrow[u, "(\lambda B)^\perp"] & GB\rangle
\end{tikzcd} \tag{39}$$

Since the companions of $\mathsf{C}F$ and $\mathsf{C}G$ are identities, the condition (27) applied on $\mathsf{C}\lambda$ boils down to the fact that the 2-cell $(\mathsf{C}\lambda)\langle A, f, B\rangle^*$ is the inverse of $(\mathsf{C}\lambda\langle A, f, B\rangle)^*$.

This completes the definition of $\mathsf{C} : \mathsf{SA} \to \mathsf{DU}$.

Proposition 3.1. The functor $\mathsf{C} : \mathsf{SA} \to \mathsf{DU}$ is right adjoint to the inclusion $\mathsf{U} : \mathsf{DU} \hookrightarrow \mathsf{SA}$.

Proof. The components of the counit $\mathsf{e} : \mathsf{UC} \to \mathsf{Id}$ and of the unit $\mathsf{h} : \mathsf{Id} \to \mathsf{CU}$ of the adjunction $\mathsf{U} \dashv \mathsf{C}$ are obtained from the functors

$$\begin{aligned} E : \mathsf{C}\mathcal{V} \to \mathcal{V} \quad &: \quad \langle A, A \stackrel{f}{\to} B^\perp, B\rangle \longmapsto B \text{ and} && (40)\\ H : \mathcal{V} \to \mathsf{C}\mathcal{V} \quad &: \quad A \longmapsto \langle A^\perp, A^\perp \stackrel{\mathrm{id}}{\to} A^\perp, A\rangle. && (41)\end{aligned}$$

For any self-adjunction $\mathcal{V}$, each of these functors underlies an SA-morphism. The former is accompanied by the natural family

$$\epsilon\langle A, A \stackrel{f}{\to} B^\perp, B\rangle \quad = \quad f : A \to B^\perp, \tag{42}$$

which has already appeared in (10); the latter by $\chi A = \langle \eta, \mathrm{id}\rangle$.

$$\begin{tikzcd} H(A^\perp) \arrow[d, "\chi A"'] & = & \langle A^{\perp\perp}, & A^{\perp\perp} \arrow[r, "\mathrm{id}"] & A^{\perp\perp}, & A^\perp\rangle \arrow[d, "\mathrm{id}"] \\ (HA)^* & = & \langle A, \arrow[u, "\eta"] & A \arrow[r, "\eta"'] \arrow[u, "\eta"] & A^{\perp\perp}, \arrow[u, "\mathrm{id}"'] & A^\perp\rangle \end{tikzcd} \tag{43}$$

Note that $\mathcal{V}$ is a duality if and only if χ is an isomorphism, which makes H into a duality morphism. The self-adjunction morphisms $\mathsf{e}_\mathcal{V} = E : \mathsf{C}\mathcal{V} \to \mathcal{V}$ thus form a natural transformation $\mathsf{e} : \mathsf{UC} \to \mathsf{Id}$ in SA, while the duality morphisms $\mathsf{h}_\mathcal{D} = H : \mathcal{D} \to \mathsf{C}\mathcal{D}$ form a natural transformation $\mathsf{h} : \mathsf{Id} \to \mathsf{CU}$. The adjunction identity $\mathsf{eU} \circ \mathsf{Uh} = \mathsf{id}_\mathsf{U}$ is immediate, while $\mathsf{Ce} \circ \mathsf{hC} = \mathsf{id}_\mathsf{C}$ boils down to expanding (34) for e. $\square$

Definitions (40–43) also yield the data of the comonad $\mathsf{G} = \mathsf{U} \cdot \mathsf{C} : \mathsf{SA} \to \mathsf{SA}$, namely

$$\begin{aligned} \mathsf{e}_\mathcal{V} &= E : \mathsf{G}\mathcal{V} \to \mathcal{V} && (44)\\ \mathsf{d}_\mathcal{V} = \mathsf{Uh}_{\mathsf{C}\mathcal{V}} &= H : \mathsf{G}\mathcal{V} \to \mathsf{G}^2\mathcal{V} && (45)\end{aligned}$$

Proposition 3.2. The inclusion $\mathsf{U} : \mathsf{DU} \hookrightarrow \mathsf{SA}$ is comonadic.

Proof. The comparison functor for the comonad G is

$$\mathsf{H} : \mathsf{DU} \to \mathsf{SA} \quad : \quad \mathcal{D} \longmapsto \Big(H : \mathcal{D} \to \mathsf{G}\mathcal{D}\Big) \tag{46}$$

where H is the SA-morphism defined by (41) and (43). This is a G-coalgebra if and only if $\mathcal{D}$ is a duality. Indeed, the triangle from (11) commutes always, and we have with $\zeta = \mathrm{id}$. The square can be filled with a 2-cell $\theta : \mathsf{d} \cdot H \to \mathsf{G}H \cdot H$, the components of which are η and the identities. This 2-cell will be invertible if and only if η is, i.e. if and only if $\mathcal{D}$ is a duality.

Similarly, a square realizing a morphism $\mathsf{H}\mathcal{V} \to \mathsf{H}\mathcal{U}$ as on (12) can always be filled with a 2-cell $\phi = \langle \varphi, \mathrm{id} \rangle$

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "F"] \arrow[d, "H"'] & \mathcal{U} \arrow[d, "H"] \arrow[dl, Rightarrow, shorten=10pt, "\phi"'] \\
\mathsf{G}\mathcal{V} \arrow[r, "\mathsf{G}F"'] & \mathsf{G}\mathcal{U}
\end{tikzcd} \tag{47}$$

where φ is the companion of the SA-morphism F. ϕ will thus be invertible if and only if φ is; i.e., F will be a coalgebra homomorphism if and only if it is a DU-morphism.
Since the 2-cells of DU and of SA coincide, we conclude that the comparison functor H is full and faithful. It remains to be shown that it is essentially surjective.
So let $K : \mathcal{V} \to \mathsf{G}\mathcal{V}$ be an arbitrary G-coalgebra. Recalling that the underlying category of $\mathsf{G}\mathcal{V}$ is $(\mathcal{V}/\perp)^{op}$, let us decompose (as in the proof of 2.1)

$$KA \quad = \quad \langle K_0A,\ K_0A \xrightarrow{kA} (K_1A)^{\perp},\ K_1A \rangle, \tag{48}$$

where where $K_0 : \mathcal{V} \to \mathcal{V}^{op}$ and $K_1 : \mathcal{V} \to \mathcal{V}$ are functors, and $k : K_0 \to \perp^{op} K_1$ is a natural transformation. Since (40) immediately yields $EK = K_1$, the filler ζ of the counit condition on K yields an isomorphism

$$j : \mathrm{Id} \overset{\sim}{\to} K_1. \tag{49}$$

In a moment, we shall prove that $k : K_0 \to \perp^{op} K_1$ is an isomorphism too — which gives the isomorphism

$$i = j^{\perp} \circ k : K_0 \overset{\sim}{\to} \perp K_1 \overset{\sim}{\to} \perp^{op}. \tag{50}$$

Together, such is and js now form a natural isomorphism $\iota : H \overset{\sim}{\to} K$.

$$\begin{tikzcd}
HA \arrow[d, "\iota A"'] & = & \langle A^{\perp}, & A^{\perp} \arrow[r, "\mathrm{id}"] & A^{\perp}, & A\rangle \arrow[d, "jA"] \\
KA & = & \langle K_0A, \arrow[u, "iA"'] & K_0A \arrow[r, "kA"'] \arrow[u, "iA"'] & (K_1A)^{\perp}, \arrow[u, "(jA)^{\perp}"'] & K_1A\rangle
\end{tikzcd} \tag{51}$$

It is not hard to check that ι is a 2-cell in SA. It accompanies the identity on $\mathcal{V}$ as a coalgebra isomorphism between H and K.

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\mathrm{Id}"] \arrow[d, "H"'] \arrow[dr, phantom, "\overset{\iota}{\cong}"] & \mathcal{V} \arrow[d, "K"] \\
\mathsf{G}\mathcal{V} \arrow[r, "\mathsf{G}(\mathrm{Id})"'] & \mathsf{G}\mathcal{V}
\end{tikzcd} \tag{52}$$

This 2-cell is coherent, in the sense of (Street 1974; Zöberlein 1976), with the companions θ, ζ of the coalgebras H and K — since it has been derived from them.

In this way, an arbitrary coalgebra $K : \mathcal{V} \to \mathsf{G}\mathcal{V}$ is shown to be isomorphic with the coalgebra $\mathsf{H}\mathcal{V} = H : \mathcal{V} \to \mathsf{G}\mathcal{V}$. This means that H is essentially surjective.
To complete the proof, we must fill the gap left behind: prove that $k : K_0 \to \perp^{op} K_1$ is an iso. As in the proof of 2.1, the task is simplified by transferring K_1 along $j : \mathrm{Id} \overset{\sim}{\to} K_1$, which makes it into the identity functor, while the transformation k becomes $K_0 \to \perp^{op}$. The companion κ of the SA-morphism K is now

$$\begin{tikzcd}
K(A^\perp) \arrow[d, "\kappa A"'] & = & \langle K_0(A^\perp), & K_0(A^\perp) \arrow[r, "k(A^\perp)"] & A^{\perp\perp}, & A^\perp\rangle \arrow[d, "\kappa_1 A"] \\
(KA)^\perp & = & \langle A, \arrow[u, "\kappa_0 A"] & A \arrow[u, "\kappa_0 A"] \arrow[r, "(kA)^*"'] & (K_0A)^\perp, \arrow[u, "(\kappa_1 A)^\perp"'] & K_0A\rangle
\end{tikzcd} \tag{53}$$

Again as in 2.1, the companion of $\mathsf{e} \cdot K$ is $\epsilon K \circ E\kappa = k \circ \kappa_1$. By the counit condition, this natural transformation must be an iso, which means that k is a split epi. Indeed, just like (cf. (18)) led to (18), condition (27), applied to $\zeta : \mathsf{Id} \to \mathsf{e} \cdot K$, now yields

$$(\zeta A)^\perp \circ k \circ \kappa_1 \circ \zeta(A^\perp) \quad = \quad \mathrm{id}_{A^\perp}. \tag{54}$$

which is
To prove that k is a monic as well, we use the chain condition, i.e. the natural isomorphism $\theta : \mathsf{C}K \cdot K \longrightarrow \mathsf{d}_\mathcal{V} \cdot K$. Its components are in $\mathsf{G}^2\mathcal{V}$, a double comma category again. The objects are thus in the form $\langle A, f, B\rangle$, where $A = \langle A_0, a, A_1\rangle$ and $B = \langle B_0, b, B_1\rangle$ are objects and $f = \langle f_0, f_1\rangle : A \to B^*$ an arrow from $\mathsf{G}\mathcal{V} = \big(\mathcal{V}/\perp\big)^{op}$. To reach the required conclusion, it suffices to calculate and compare the components f_1 of $\mathsf{C}K \cdot K(A)$ and of $\mathsf{d}_\mathcal{V} \cdot K(A)$. The latter f_1 is the identity on K_0A. The former is $\kappa_1 A \circ kA : K_0A \to A^\perp \to K_0A$. Since the chain condition, i.e. the 2-cell θ, makes these two arrows isomorphic, kA must be a split monic. But we already know that it is a split epi — so it must be an iso. Hence the result: K is isomorphic to H. The comparison functor H is an equivalence. □

A KZ-*comonad?* No, $\mathsf{G} : \mathsf{SA} \to \mathsf{SA}$ is not a KZ-comonad either. Since all 2-cells in SA are invertible, an SA-morphism can have an adjoint only if the underlying functor is an equivalence. But $\mathsf{G}\mathcal{V}$ is equivalent to $\mathcal{V}$ if and only if $\mathcal{V}$ is a groupoid. In general, thus, a coalgebra $K : \mathcal{V} \to \mathsf{G}\mathcal{V}$ is not adjoint to the counit $\mathsf{e}_\mathcal{V} : \mathsf{G}\mathcal{V} \to \mathcal{V}$.

4. The Chu comonad

In this section we finally focus on self-adjunctions induced by internal homming (2) in autonomous categories. They form a category $\mathsf{AU}_\perp$. Its morphisms preserve the autonomous structure up to specified natural transformations, just like the SA-morphisms preserved $\perp$. More than this lax preservation property would not be preserved by the arrow part of the comma construction.

On the other hand, the self-dual autonomous categories form the category AU_*. There are forgetful functors $\mathsf{AU}_\perp \to \mathsf{SA}$ and $\mathsf{AU}_* \to \mathsf{DU}$, and proposition 3.2 will be lifted along them.

4.1. $\perp$-*Autonomous categories*

An object $\mathcal{V}$ of $\mathsf{AU}_\perp$ is an autonomous category with a distinguished object $\perp$. The autonomous, or symmetric monoidal closed structure, consists of a tensor $\otimes$, a unit $\top$, and a cotensor $\multimap$, tied together with a standard set of coherent natural transformations (Kelly 1982, sec. 1.4–1.5). This structure does not restrict the choice of $\perp$ in any way, and this can be any object of $\mathcal{V}$. It just induces and represents a self-adjunction on $\mathcal{V}$.

For reasons which will become clear in 4.3.1, we shall consider only those $\perp$-autonomous categories $\mathcal{V} \in \mathsf{AU}_\perp$ which have pullbacks, as well as pushouts. However, note that these limits are not treated as structure; their existence is just a needed property. The morphisms will thus not be required to preserve them.

An $\mathsf{AU}_\perp$-morphism from $\mathcal{V}$ to $\mathcal{U}$ is a functor $F : \mathcal{V} \to \mathcal{U}$ accompanied with the following natural transformations.

$$
\begin{tikzcd}
 & 1 \arrow[dl, "\top"'] \arrow[dr, "\top"] \arrow[d, phantom, "\Downarrow \phi_\top"] & \\
\mathcal{V} \arrow[rr, "F"'] & {} & \mathcal{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[dl, "\perp"'] \arrow[dr, "\perp"] \arrow[d, phantom, "\Uparrow \phi_\perp"] & \\
\mathcal{V} \arrow[rr, "F"'] & {} & \mathcal{U}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{V} \times \mathcal{V} \arrow[r, "F \times F"'] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\swarrow \phi_\otimes"] & \mathcal{U} \times \mathcal{U} \arrow[d, "\otimes"] \\
\mathcal{V} \arrow[r, "F"'] & \mathcal{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}^{op} \times \mathcal{V} \arrow[r, "F^{op} \times F"'] \arrow[d, "\multimap"'] \arrow[dr, phantom, "\nearrow \phi_\multimap"] & \mathcal{U}^{op} \times \mathcal{U} \arrow[d, "\multimap"] \\
\mathcal{V} \arrow[r, "F"'] & \mathcal{U}
\end{tikzcd}
\tag{55}
$$

These companions must be coherent with the autonomous structure. To state this requirement precisely, let us consider a diagram Δ, composed of some of the transformations ϕ from (55), together with some of those coming from the autonomous structure, or their F-images. Let $|\Delta|$ be the diagram obtained from Δ by erasing F and replacing each occurrence of ϕ with the identity. If Δ involved functors with at most n arguments, then $|\Delta|$ will be a diagram in the free autonomous category with $n + 1$ generators. The additional generator plays the role of $\perp$.

The coherence condition on ϕ is that

$$\Delta \text{ must commute whenever } |\Delta| \text{ commutes.}$$

An explicit set of coherence conditions for ϕ can thus be derived from any set of coherence conditions for the autonomous structure.

Proving the compositionality of $\mathsf{AU}_\perp$-morphisms is a routine exercise. The companions are defined by composing 2-cells (55).

If $F, G : \mathcal{V} \to \mathcal{U}$ are two $\mathsf{AU}_\perp$-morphisms, a 2-cell λ from F to G in $\mathsf{AU}_\perp$ is a natural isomorphism $\lambda : F \to G$ which yields yields the companions γ of G when pasted in (55) wherever possible.

This completes the definition of the category $\mathsf{AU}_\perp$. Each of its objects is an autonomous category *and* a self-adjunction, with $⫫$ induced by homming into $\perp$. Its morphisms lax preserve the autonomous structure *and* $⫫$. So they are also self-adjunction morphisms, with companions derived from the right-hand side of (55). Finally, the 2-cells of $\mathsf{AU}_\perp$ also qualify as 2-cells of SA. Hence the forgetful functor $\mathsf{AU}_\perp \to \mathsf{SA}$.

4.2. $*$-*Autonomous categories*

By definition, a category $\mathcal{D}$ is $*$-autonomous if it is autonomous and self-dual. In fact, it is enough to have the symmetric monoidal structure $\otimes, \top$, and an equivalence $* : \mathcal{D}^{op} \to \mathcal{D}$, tied together by natural transformations saying that, for every A

$$A \multimap (-) \quad = \quad \big(A \otimes (-)^*\big)^* \tag{56}$$

is right adjoint to $A \otimes (-)$. Hence the closed structure. Categories with these data form AU_*.

A morphism between the $*$-autonomous categories $\mathcal{D}$ and $\mathcal{E}$ will be a functor $G : \mathcal{D} \to \mathcal{E}$, accompanied with the data from the left-hand side of (55), plus (29)

$$\begin{aligned} \gamma_\top \quad &: \quad \top \longrightarrow G\top \\ \gamma_\otimes AB \quad &: \quad G(A) \otimes G(B) \longrightarrow G(A \otimes B) \\ \gamma_* A \quad &: \quad G(A^*) \stackrel{\sim}{\longrightarrow} (GA)^* \end{aligned} \tag{57}$$

The coherence condition can be stated as for the $\mathsf{AU}_\perp$-morphisms above. The 2-cells of AU_* will again be just the natural isomorphisms, coherent with all the involved companions. Hence the category AU_*, with the obvious forgetful functor $\mathsf{AU}_* \to \mathsf{DU}$.

On the other hand, there is a functor $\mathsf{U} : \mathsf{AU}_* \to \mathsf{AU}_\perp$, injective on objects, faithful on morphisms, full and faithful on 2-cells. It could even be made into an inclusion by adding some redundant data in the definition of AU_*.

First of all, each $*$-autonomous category can be viewed as an object of $\mathsf{AU}_\perp$ — with the autonomous structure supplied by (56), and $\perp$ defined to be $\top^*$. Furthermore, each AU_*-morphism G yields an $\mathsf{AU}_\perp$-morphism with the same underlying functor. The companion $\gamma_\perp$ is derived from $\gamma_\top$ and γ_*, using $\perp = \top^*$, while $\gamma_\multimap$ is derived from $\gamma_\otimes$ and γ_*, using (56). The 2-cells are then found to coincide.

4.3. *The comonad*

The functor $\mathsf{Chu} : \mathsf{AU}_\perp \longrightarrow \mathsf{AU}_*$ will be obtained by lifting $\mathsf{C} : \mathsf{SA} \longrightarrow \mathsf{DU}$ along the forgetful functors $\mathsf{AU}_\perp \to \mathsf{SA}$ and $\mathsf{AU}_* \to \mathsf{DU}$. In other words, to get Chu from C, one just adds the autonomous structure.

4.3.1. *Object part.* The $*$-autonomous category $\mathsf{Chu}(\mathcal{V})$, associated with the $\perp$-autonomous category $\mathcal{V} \in \mathsf{AU}_\perp$ is thus the comma $(\mathcal{V}/⫫)^{op}$ again. The duality $*$ is as in (31–33). Extending the autonomous structure from $\mathcal{V}$ to $\mathsf{Chu}(\mathcal{V})$ is more subtle, though. This is perhaps the most important contribution of (Chu 1979).

First of all, the duality $*$ on any $*$-autonomous category $\mathcal{D}$, can be viewed as a "De Morgan switch" between the autonomous structure $(\top, \otimes, \multimap)$ of $\mathcal{D}$ and the autonomous structure $(\bot, \oplus, \multimapinv)$ of $\mathcal{D}^{op}$. Thus, either of the tensors or cotensors, with either of the units — determines all.

If $\mathcal{D} = \mathsf{Chu}(\mathcal{V})$, the cotensor $\multimapinv$ must internalize the hom-sets of the comma category $\mathcal{D}^{op} = \mathcal{V}/\!\!\perp\!\!\!\perp$. For any objects $X = \langle A, f, B\rangle$ and $Y = \langle C, g, D\rangle$ from this category, $Y \multimapinv X$ should thus represent *"the set of all pairs $u : A \to C$ and $v : D \to B$ such that $g \circ u = v^\bot \circ f$"* (3). Using the closed structure of $\mathcal{V}$, this set is encoded as the pullback

$$\begin{tikzcd}
P \arrow[rr] \arrow[ddd] \arrow[ddr, "h"] & & D \multimap B \arrow[d] \\
& & B^\bot \multimap D^\bot \arrow[dd, "f \multimap D^\bot"] \\
& (A \otimes D)^\bot & \\
A \multimap C \arrow[rr, "A \multimap g"'] & & A \multimap D^\bot \arrow[ul, "\sim"]
\end{tikzcd} \tag{58}$$

which leads to the definition:

$$Y \multimapinv X \quad = \quad \langle P, P \xrightarrow{h} (A \otimes D)^\bot, A \otimes D\rangle. \tag{59}$$

The corresponding unit $\tilde{\bot}$ must be $\langle \top, \eta_\top, \bot\rangle$, since only this object ensures the natural correspondence

$$\mathsf{Chu}(\mathcal{V})\left(Y \multimapinv X, \tilde{\bot}\right) \quad \cong \mathsf{Chu}(\mathcal{V})(Y, X). \tag{60}$$

The autonomous structure of $\mathsf{Chu}(\mathcal{V})$ is now derived as follows:

$$\tilde{\top} \quad = \quad \tilde{\bot}^* \quad \left(= \langle \bot, \bot \xrightarrow{\mathrm{id}} \bot, \top\rangle\right) \tag{61}$$

$$X \otimes Y \quad = \quad Y \multimapinv X^* \tag{62}$$

$$X \multimap Y \quad = \quad \left(Y^* \multimapinv X^*\right)^*. \tag{63}$$

Spelling out the construction of $X \otimes Y = \langle P', h', B \otimes D\rangle$, one gets the pullback

$$\begin{tikzcd}
P' \arrow[rr] \arrow[dd] \arrow[ddrr, "h'"] & & D \multimap A \arrow[d, "D \multimap f"] \\
& & D \multimap B^\bot \arrow[d, "\sim"] \\
B \multimap C \arrow[r, "B \multimap g"'] & B \multimap D^\bot \arrow[r, "\sim"'] & (B \otimes D)^\bot
\end{tikzcd} \tag{64}$$

which is actually simpler to memorize than (58), although it does not appear to be as directly motivated.

In principle, there is no reason why the described $*$-autonomous structure of $\mathsf{Chu}(\mathcal{V})$ would be the only one. In order to establish Chu as a functor well-defined in itself, one is thus led to look for a sense in which this structure would be canonical, i.e. completely determined by the autonomous structure of $\mathcal{V}$, together with some preservation requirements. For instance, note that both $E : \mathsf{Chu}(\mathcal{V}) \longrightarrow \mathcal{V}$ (40) and $H : \mathcal{V} \longrightarrow \mathsf{Chu}(\mathcal{V})$ (41) preserve the monoidal structure defined above. However, requiring that these functors are monoidal does not pin down the structure of $\mathsf{Chu}(\mathcal{V})$. A different structure, still satisfying this requirement, can be obtained if the the object P, defined on (58), is replaced in the definition of $Y \multimapinv X$ by any subobject $R \hookrightarrow P$ that contains all global points $\top \to P$. This indeed ensures that condition (60) still holds, with the same $\tilde{\bot}$.

To enforce definition (59), one needs to take into account a more general condition than (60), namely

$$\mathsf{Chu}(\mathcal{V})\left(Y \multimapinv X,\ (HA)^*\right) \ \cong \mathsf{Chu}(\mathcal{V})\left(Y \otimes HA,\ X\right), \tag{65}$$

and to require not just that H preserves $\top$, but that for every $A \in \mathcal{V}$, the functor $(-) \otimes HA = (-) \multimapinv (HA)^*$ is defined componentwise, i.e. that

$$Y \multimapinv (HA)^* \quad = \quad \langle A \multimap C,\ A \multimap C \stackrel{A \multimap g}{\longrightarrow} A \multimap D^\perp \cong (A \otimes D)^\perp,\ A \otimes D \rangle, \tag{66}$$

holds for any $Y = \langle C, g, D \rangle$. It is easy to see that this is satisfied by definition (59). The other way around, using (65), one readily shows that (66) implies (59).

In this way, $\mathsf{Chu}(\mathcal{V})$, as a $*$-autonomous category, is completely determined.

Remark. So far, we have actually shown that the functor $\mathsf{C} : \mathsf{SA} \to \mathsf{DU}$, applied to a $\bot$-autonomous category with pullbacks, yields a $*$-autonomous category. However, to get a $*$-autonomous category *with pullbacks* again, one must start from a $\bot$-autonomous category which also has *pushouts*, and not only pullbacks. This is a consequence of the lemma in the appendix.

4.3.2. *Arrow part.* Given an $\mathsf{AU}_\bot$-morphism $F : \mathcal{V} \to \mathcal{U}$, the underlying functor of $G = \mathsf{Chu}(F) : \mathsf{Chu}(\mathcal{V}) \longrightarrow \mathsf{Chu}(\mathcal{U})$ is defined as on (34–35). The transformation $F(B^\perp) \stackrel{\varphi}{\to} (FB)^\perp$, used there, is now obtained as the composite

$$F(B \multimap \bot) \stackrel{\phi_{\multimap}}{\longrightarrow} (FB \multimap F\bot) \stackrel{\phi_\bot}{\longrightarrow} (FB \multimap \bot). \tag{67}$$

The coherence of (55) ensures that this transformation satisfies (25), which implies, as in (38), that the defined functor strictly preserves $*$. The companion γ_* of $\mathsf{Chu}(F)$ is thus the identity.

On the other hand, $\gamma_\top : \top \to G\top$ is just $\langle \phi_\bot, \phi_\top \rangle$. Indeed, the coherence of ϕ guarantees that the square on

$$\begin{tikzcd} \langle \bot, & \bot \arrow[r, "\mathrm{id}"] & \bot, & \top \rangle \arrow[d, "\phi_\top"] \\ \langle F\bot, \arrow[u, "\phi_\bot"] & F\bot \arrow[u, "\phi_\bot"] \arrow[r, "\varphi"'] & (F\top)^\perp, \arrow[u, "\phi_\top^\perp"'] & F\top \rangle \end{tikzcd} \tag{68}$$

commutes. Finally $\gamma_\otimes : GX \otimes GY \longrightarrow G(X \otimes Y)$ is $\langle q, \phi_\otimes \rangle$, i.e.

$$
\begin{tikzcd}
\langle P'', & P'' \arrow[rr, "h''"] & & (FB \otimes FD)^\perp, & FB \otimes FD \rangle \arrow[d, "\phi_\otimes"] \\
\langle FP', \arrow[u, "q"] & FP' \arrow[u, "q"] \arrow[r, "Fh'"'] & F\left((B \otimes D)^\perp\right) \arrow[r, "\varphi"'] & \left(F(B \otimes D)\right)^\perp, \arrow[u, "\phi_\otimes^\perp"'] & F(B \otimes D) \rangle
\end{tikzcd}
\tag{69}
$$

where q is defined on the following diagram.

$$
\begin{tikzcd}
FP' \arrow[rrrr] \arrow[dr, dashed, "q"] \arrow[dddd] & & & & F(D \multimap A) \arrow[dl, "\phi_\multimap"'] \arrow[dddd] \\
& P'' \arrow[rr] \arrow[d] \arrow[drr, "h''"] & & FD \multimap FA \arrow[d] & \\
& FB \multimap FC \arrow[rr] & & (FB \otimes FD)^\perp & \\
& & & \left(F(B \otimes D)\right)^\perp \arrow[u, "\phi_\otimes^\perp"'] & \\
F(B \multimap C) \arrow[uur, "\phi_\multimap"] \arrow[rrrr] & & & & F\left((B \otimes D)^\perp\right) \arrow[ul, "\varphi"']
\end{tikzcd}
\tag{70}
$$

The larger square is just the F-image of (64), while the smaller one is (64) instanciated with the F-images of X and Y. The latter is thus the pullback square which defines $GX \otimes GY$. Since the bottom and the right-hand side trapezoids commute by coherence, the arrow from FP' to $(FB \otimes FD)^\perp$ *via* $FD \multimap FA$ must be equal with the one *via* $FB \multimap FC$. Hence the unique arrow q, making (70) commute.

The coherence of the constructed companions γ follows directly from the coherence of ϕ.

The fact that the 2-cell $\mathsf{Chu}(\lambda) : \mathsf{Chu}(F) \to \mathsf{Chu}(F')$, defined as on (39), will be coherent with respect to such companions of $\mathsf{Chu}(F)$ and of $\mathsf{Chu}(F')$ follows from the coherence of $\lambda : F \to F'$ with respect to the companions of F and F'.

4.3.3. *Couniversality of* Chu. To prove that Chu is right adjoint to $\mathsf{U} : \mathsf{AU}_* \to \mathsf{AU}_\perp$, i.e. to lift proposition 3.1, one needs to show that the functors $E : \mathsf{Chu}(\mathcal{V}) \to \mathcal{V}$ and $H : \mathcal{V} \to \mathsf{Chu}(\mathcal{V})$, defined as before, induce $\mathsf{AU}_\perp$-morphisms, and that the latter preserves $*$ up to iso when $\mathcal{V}$ is $*$-autonomous. These morphisms form the units of the adjunction $\mathsf{U} \dashv \mathsf{Chu}$.

It is immediate from the definitions that both E and H strictly preserve $\top$ and $\otimes$. Moreover, E preserves $\perp$. So we only need to spell out

$$\epsilon_\multimap XY \quad : \quad E(X \multimap Y) \longrightarrow (EX \multimap EY)$$

$$
\begin{aligned}
\chi_{\multimap}AB &: H(A \multimap B) \longrightarrow (HA \multimap HB) \text{ and}\\
\chi_{\perp} &: H\perp \longrightarrow \perp
\end{aligned}
$$

for arbitrary A, B from $\mathcal{V}$ and $X = \langle A, f, B\rangle$ and $Y = \langle C, g, D\rangle$ from $\mathsf{Chu}(\mathcal{V})$. The remaining companions of E and H will all be identities.

To calculate $X \multimap Y = (X \otimes Y^*)^*$, instanciate (64) again.

$$
\begin{tikzcd}
Q \arrow[r] \arrow[d, "\epsilon_{\multimap}"'] \arrow[dr, "k"] & C \multimap A \arrow[d] \\
B \multimap D \arrow[r] & (B \otimes C)^{\perp}
\end{tikzcd}
\tag{71}
$$

Hence $X \otimes Y^* = \langle Q, k, B \otimes C\rangle$. The transposition now yields $X \multimap Y = \langle B \otimes C, k^*, Q\rangle$, and thus $E(X \multimap Y) = Q$. Since $EX = B$ and $EY = D$, the $\multimap$-companion of E is actually the arrow $Q \longrightarrow (B \multimap D)$ from (71). The universality of its construction ensures the coherence.

Similarly, $HA \multimap HB$ is obtained by transposing the pullback

$$
\begin{tikzcd}
A \multimap B \arrow[rr] \arrow[d, "="'] \arrow[drr, "m"] & & B^{\perp} \multimap A^{\perp} \arrow[d, "\sim"] \\
A \multimap B \arrow[r, "A \multimap \eta"'] & A \multimap B^{\perp\perp} \arrow[r, "\sim"'] & \left(A \otimes B^{\perp}\right)^{\perp}
\end{tikzcd}
\tag{72}
$$

which defines $HA \otimes HB^* = \langle A \multimap B, m, A \otimes B^{\perp}\rangle$. Hence $\chi_{\multimap} : H(A \multimap B) \longrightarrow (HA \multimap HB)$

$$
\begin{tikzcd}
\langle (A \multimap B)^{\perp}, & (A \multimap B)^{\perp} \arrow[r, "\mathrm{id}"] & (A \multimap B)^{\perp}, & A \multimap B\rangle \arrow[d, "\mathrm{id}"] \\
\langle A \otimes B^{\perp}, \arrow[u, "m^*"] & A \otimes B^{\perp} \arrow[u, "m^*"] \arrow[r, "m^*"'] & (A \multimap B)^{\perp}, \arrow[u, "\mathrm{id}"'] & A \multimap B\rangle
\end{tikzcd}
\tag{73}
$$

Finally, the $\chi_{\perp} : H\perp \rightarrow \perp$ is just the $\top$-component of (43).

$$
\begin{tikzcd}
\langle \perp^{\perp}, & \perp^{\perp} \arrow[r, "\mathrm{id}"] & \perp^{\perp}, & \perp\rangle \arrow[d, "\mathrm{id}"] \\
\langle \top, \arrow[u, "\eta"] & \top \arrow[u, "\eta"] \arrow[r, "\eta"'] & \perp^{\perp}, \arrow[u, "\mathrm{id}"'] & \perp\rangle
\end{tikzcd}
\tag{74}
$$

The coherence of these companions of H is a direct consequence of the way in which they are derived from the closed structure of $\mathcal{V}$. On the other hand, note that $\mathcal{V}$ is a $*$-autonomous category if and only if all $\eta : B \to B^{\perp\perp}$ are isomorphisms, i.e. if and only if $\chi_{\multimap}$ and $\chi_{\perp}$ are isomorphisms. Thus, $\mathcal{V}$ is $*$-autonomous if and only if $H : \mathcal{V} \longrightarrow \mathsf{Chu}(\mathcal{V})$ is an AU_*-morphism.

Showing that E and H realize the adjunction $\mathsf{U} \dashv \mathsf{Chu}$ and that U is comonadic is exactly the same as in 3.1 and 3.2. The demonstrated couniversality thus does not essentially depend on the autonomous structure. It is therefore even more interesting that the involved constructions preserve the autonomous structure, when present, in such a remarkable way, as shown by the Chu construction.

Theorem 4.1. The functor $\mathsf{Chu} : \mathsf{AU}_\perp \longrightarrow \mathsf{AU}_*$ is right adjoint to the forgetful functor $\mathsf{U} : \mathsf{AU}_* \to \mathsf{AU}_\perp$ (cf. sec. 4.2). This functor is comonadic: the induced coalgebras and homomorphisms exactly correspond to $*$-autonomous categories and their morphisms.

Remark. With more work, a similar result could be proved for Hyland-de Paiva's Dialectica categories (de Paiva 1989). Just like the Chu categories, they can be reduced to the comma construction, only this time poset-enriched.

Appendix A. Limits and colimits in a Chu category

The limits in $\mathsf{Chu}(\mathcal{V})$ are constructed using both the limits and the colimits in $\mathcal{V}$. In view of the self-duality of $\mathsf{Chu}(\mathcal{V})$, the same must be true for the colimits in it. The other way around, the limits and the colimits in $\mathcal{V}$ can be reconstructed from either limits or colimits of $\mathsf{Chu}(\mathcal{V})$.

Lemma. Let $\mathcal{V}$ be a category, $\perp\!\!\!\perp : \mathcal{V}^{op} \to \mathcal{V}$ a functor, self-adjoint on the right, and Γ a class of diagram schemes. The following conditions are equivalent:

(a). $\mathcal{V}$ has Γ-limits and Γ-colimits.
(b). $\mathcal{V}/\perp\!\!\!\perp$ has Γ-limits.
(c). $\mathcal{V}/\perp\!\!\!\perp$ has Γ-colimits.

Proof. (a)⇒(b) is based on the fact that the functor $\perp\!\!\!\perp$ preserves colimits. For instance, since $(B+D)^\perp = B^\perp \times D^\perp$, the binary product in $\mathcal{V}/\perp\!\!\!\perp$ can be defined:

$$\langle A, f, B\rangle \times \langle C, g, D\rangle = \langle A \times C, A \times C \xrightarrow{f\times g} B^\perp \times D^\perp, B + D\rangle. \tag{75}$$

Since $\mathcal{V}/\perp\!\!\!\perp$ is self-dual, (b)⇔(c) is obvious.
To complete the proof, it suffices to show (b)∧(c)⇒(a). For this, we use the coreflection $H \dashv E : \left(\mathcal{V}/\perp\!\!\!\perp\right)^{op} \to \mathcal{V}$ from (40–41) (Of course, the coreflection of $\mathcal{V}$ in $\mathcal{V}/\perp\!\!\!\perp$ would do as well; but $H \dashv E$ has already been spelled out.) If $\Delta : I \to \mathcal{V}$ is a diagram, with $I \in \Gamma$, and if L and C are respectively a limit and a colimit of the diagram $H\Delta$ in $\left(\mathcal{V}/\perp\!\!\!\perp\right)^{op}$, then EL and EC will be respectively a limit and a colimit of Δ in $\mathcal{V}$. The former is obvious, since E must preserve limits. The latter follows from the fact that the inclusion of a (co)reflective subcategory creates (co)limits, which is easy to check and surely belongs to folklore. □

References

Appelgate, H. and Tierney, M. (1969) Categories with models. In B. Eckmann (editor), *Seminar on Triples, Categories and Categorical Homology Theory*, Lecture Notes in Mathematics **80**, 156–244. Springer-Verlag.

Barr, M. (1979) *∗-Autonomous Categories*, Lecture Notes in Mathematics **752**. Springer-Verlag.

Barr, M. (1990) Accessible categories and models of linear logic. *J. Pure Appl. Algebra* **69**, 219–232.

Barr, M. (1991) $*$-Autonomous categories and linear logic. *Math. Structures Comput. Sci.* **1**(2), 159–178.

Blackwell, R., Kelly, G.M. and Power, A.J. (1989) Two-dimensional monad theory. *J. Pure Appl. Algebra* **59**, 1–41.

Brown, C. and Gurr, D. (1991) A categorical linear framework for Petri nets. In *Proc. 5th Ann. Symp. on Logics in Comp. Sci.*, 208–218. IEEE Computer Society Press.

Chu, P.-H. (1979) Constructing $*$-autonomous categories, Appendix in (Barr 1979), 103–138.

Glabbeek, R.J. van, and Plotkin, G.D. (1995) Configuration structures. In *Proc. 10th Symp. on Logics in Comp. Sci.*, 199–209. IEEE Computer Society Press.

Gupta, V. (1994) *Chu Spaces: A Model of Concurrency.* Thesis. Stanford University.

Gupta, V. and Pratt, V.R. (1993) Gates accept concurrent behavior. In *Proc. 34th Ann. Symp. on Foundations of Comp. Sci.*, 62–71. IEEE Computer Society Press.

Johnstone, P.T. and Moerdijk, I. (1989) Local maps of toposes. *Proc. Lond. Math. Soc., Ser. 3* **58**, 281–305

Kelley, J.L., Namioka, I. *et al.* (1963) *Linear Topological Spaces.* D. van Nostrand Comp.

Kelly, G.M. (1982) *Basic Concepts of Enriched Category Theory*, L.M.S. Lecture Notes Ser. **64**. Cambridge Univ. Press.

Kock, A. (1972–95) Monads for which structures are adjoint to units. Version 1: *Aarhus preprint* **35**(1972/73); version 2, *Aarhus preprint* (1992); version 3, to appear in *J. Pure Appl. Algebra* (1995), available by FTP from `theory.doc.ic.ac.uk`)

Lafont, Y. and Streicher, T. (1991) Games semantics for linear logic. In *Proc. of the 6th Symp. on Logics in Comp. Sci.*, 43–50. IEEE Computer Society Press.

Lawvere, F.W. *Functorial Semantics of Algebraic Theories.* Thesis. Columbia University.

Mackey, G.W. (1945) On infinite-dimensional linear spaces. *Trans. Amer. Math. Soc.* **57**, 155–207.

Mac Lane, S. (1971) *Categories for the Working Mathematician.* Graduate Texts in Math. **5**. Springer-Verlag.

de Paiva, V.C.V. (1989) The Dialectica categories, In Gray, J. and Scedrov, A. (editors), *Categories in Computer Science and Logic. Contemp. Math.* **92**, 47–62. Amer. Math. Soc.

Pavlović, D. (1995) Maps II: Diagram chasing in categorical proof theory. *Journal of the IGPL* **3**(7)

Pratt, V.R. (1993) Linear logic and generalized quantum mechanics. In *Proc. Workshop on Physics and Computation*, 166–180. IEEE Computer Society Press.

Pratt, V.R. (1993) The second calculus of binary relations, In *Proc. Conference on Math. Foundations of Comp. Sci.* IEEE Computer Society Press.

Pratt, V.R. (1994) Chu Spaces: Automata with quantum aspects, In *Proc. Workshop on Physics and Computation.* IEEE Computer Society Press.

Pratt, V.R. (1994) Complementarity and Uncertainty in Rational Mechanics. Course notes.

Pratt, V.R. (1995) The Stone Gamut: A coordinatization of mathematics. In: *Proc. of the 10th Symp. on Logics in Comp. Sci.*, 444–454. IEEE Computer Society Press.

Seely, R.A.G. (1989) Linear logic, $*$-autonomous categories and cofree coalgebras. In J. Gray and A. Scedrov (editors), *Categories in Computer Science and Logic. Contemp. Math.* **92**, 371–382. Amer. Math. Soc.

Street, R. (1974) Fibrations and Yoneda's lemma in a 2-category. In G.M. Kelly (editor), *Category Seminar, Sydney 1972/73.* Lecture Notes in Mathematics **420** 104–133. Springer-Verlag.

Street, R. (1980) Fibrations in bicategories. *Cahiers Top. et Géom. Dif.* **21**, 111–160

Zöberlein, V. (1976) Doctrines on 2-categories, *Math. Zeitschrift* **148**, 267–279.