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On completeness and cocompleteness in and around small categories

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Abstract

The simple connection of completeness and cocompleteness of lattices grows in categories into the Adjoint Functor Theorem. The connection of completeness and cocompleteness of Boolean algebras – even simpler – is similarly related to Paré's Theorem for toposes. We explain these relations, and then study the fibrational versions of both these theorems – for small complete categories. They can be interpreted as definability results in logic with proofs-as-constructions, and transferred to type theory.

0. Introduction

0.1. Strong constructivism and its models

Before the time of computers, it was not easy to justify the requirements of constructivism. Most mathematicians saw little reason to complicate life by rejecting the Law of Excluded Middle – and the much stronger idea of *proofs-as-constructions* really looked like a philosophical whim. However, even when it was not much practiced, constructivism was extensively studied (usually by classical means) – and this *strong constructivist* idea of proofs-as-constructions kept reappearing in various disguises: as the paradigm of *propositions-as-types* in combinatory logic [8, 16], in proof theory, in the type systems of Girard [12] and of Martin-Löf [27] ... But these logical structures were rather abstract, and did not seem to have too interesting “mathematical” models.

Computer science blew a new life in constructivism. For instance, almost all *logical frameworks* for computation – ever since the AUTOMATH-project [4] in the late

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sixties – were variations on the theme of propositions-as-types [17]. The systems of Martin-Löf [27] and of Girard [12] – the latter independently rediscovered by Reynolds [39] – dominate the landscape. It seems that nowadays, the foundations of constructive mathematics are being built as the foundations of computer science.

But even on the solid ground of computation, the idea of proofs-as-constructions remains conceptually difficult and semantically problematic. Namely, if instead of mere provability $\alpha \vdash \beta$, several different proofs-as-constructions $f: \alpha \rightarrow \beta$ can be specified for propositions-as-types α and β , the set Ω of propositions (truth values) will form a category, not just a lattice. The quantifiers over a set I will be the I -indexed sums and products in this category Ω [23]. Higher-order logic will thus require that Ω has sums and products over arbitrary sets. But now a basic exercise in category theory, due to Freyd [9, 3.D], says that a small category with all small sums or products must be a lattice – so there are *no* honest mathematical models for higher-order logic with proofs-as-constructions?!

Indeed, not in the framework of classical mathematics. A different argument, due to Reynolds [40], actually shows that already the second-order universal quantification over propositions-as-types precludes ordinary set-theoretic interpretations. But Pitts [37] has shown that *constructive* set theory of Grothendieck toposes accomodates them. However, Freyd's exercise goes through in Grothendieck toposes, so that higher-order strongly constructive logic still cannot be modelled. It requires a more constructive setting.

Hyland's effective topos [18] is the universe of sets built around Kleene's notion of *realizability*. From the outset, the partial equivalence relations on natural numbers played an important role in it. E. Moggi, a student of computer science at the time, was the first one to realize that they might form a small complete category. Within a couple of years, several proofs of this conjecture were published [5, 19, 25], emphasizing the importance of the discovery. So far, all the small complete categories known to us are based on this one. But there are plenty of them: a small complete category gives rise to others in the same way as, say, the category of sets gives rise to toposes and to algebraic categories. Moreover, some small categories with small products, but not all equalizers, have been constructed using the domain-theoretical methods [6, 20].

And now that these unicorns – the small complete categories – have been captured, the problem becomes: *How to approach them?* Ordinary category theory remains blind for their completeness. In a topos, of course, category theory can be formalized using the internal language. But a semantics of strong constructivism in terms of another formal language does not seem very informative. One would rather like to effectively calculate some (co)limits representing specific propositions-as-types. For this purpose, a small category must be *externalized* and considered as *fibred* over \mathcal{S} . But there is a price of this effectiveness: the fibrational notion completeness is essentially stronger than the internal one [41, 21, Section 4]. For instance, the category of partial equivalence relations is internally, but not fibrationally complete over the effective topos [21, Theorem 7.9 and the appendix]. We shall later show in detail that it is fibrationally complete over a suitable subcategory, which is not a topos.

0.2. Contents of the paper

We are concerned with limits and colimits as related to logical operations. In ordinary logic, they boil down to infima and suprema. Everybody knows how these are connected: the supremum of a set can be calculated as the infimum of the set of all its upper bounds. A complete lattice is always cocomplete. This is not true for categories, because they can be large, and the upper bounds of a small diagram may constitute a large diagram, with no limit. This is why the *solution set condition* must be imposed in the Adjoint Functor Theorem (another exercise of Freyd's). In small categories, however, this problem of size does not come about. *Is the completeness then equivalent to cocompleteness again?* – Not in general. In Section 3, we shall see a counterexample. Things are not as simple as the “*solution-set-condition-is-always-satisfied*” kind of intuition might suggest.

In this paper, we shall show that colimits can be derived from limits

- (I) in a small complete category \mathbb{B} (Sections 2 and 3),
- (II) in a category \mathcal{S} which contains a small complete category (Sections 4 and 5) provided that either
 - (A) \mathbb{B} is Cartesian closed, or that
 - (B) \mathcal{S} is locally Cartesian closed.

Each of the main theorems will appear in two versions, (A) and (B). Translated in type theory [30, 31], assumptions (A) and (B) roughly correspond to systems from dissertations of Girard and of Martin-Löf, respectively (enriched with the equality types). Our results are thus naturally accommodated and easily formulated in type theory. But the proofs are not easily translated: spelled out in λ -terms, our constructions seem to be growing out of proportion. Nevertheless, one may choose to regard all those fibrations and diagrams as a shorthand presentation of some type theoretical constructions.

In Section 1, we describe the ideas and tools for constructions (I) and (II): some adjoint functor theorems effective enough to be implemented in fibrations. Section 2 lifts the lattice-theoretical construction of suprema from infima to small (fibred) categories. Section 3 explains how to extend this to small *full subcategories*. (Conceptually they are, of course, a special case – but not technically). Section 4, on the other hand, generalizes Paré's [28] topos-theoretical construction of colimits from limits. We show that it goes through in every category with a small complete full subcategory, exponentiable and comprehensive. In Section 5, we extend this construction to a category containing a small complete category in general, with the same properties.

0.3. Towards relative topos

It should be stressed that these generalizations are not being spelled out for their own sake. Our goal is to make them applicable in universes with strongly constructive logic. These universes can be construed as “toposes” where the set of truth values is a nondegenerate small complete category. A concentrated research of such structures

has been initiated by the analyses of the Theory of Constructions [7, 20]. In [30, 31], it has been argued that this strong type system should be understood as a constructivist universe. In [32] the name *relative topos* has been proposed – “relative” because the family of *extents*, classified by the category of truth values, is not a canonical family any more, as the monics are in a topos. Papers [35, 36] can also be seen as a part of this project of analysing relative topos.

1. Constructing adjoint functors

1.1. Idea. Consider the following proportion:

$$\frac{\bigvee D = \bigwedge \{x \geq D\}}{\varinjlim D \sqsubset \varprojlim \{x \in \Sigma \mid D \rightarrow x\}} = \frac{\bigvee D = \bigwedge \neg \bigwedge \neg D}{\varinjlim D \sqsubset \wp(\varprojlim \wp D)} \tag{1}$$

The formula in the numerator on the left-hand side expresses the supremum of a set D in a complete (semi)lattice as the infimum of the set of all the upper bounds of D . The denominator represents the categorical generalization of this formula: the General Adjoint Functor Theorem [9, 3.J; 26, V.6; 1, 1.9]. It says that the colimit of a diagram D in a complete category \mathcal{B} can be calculated as the limit of the diagram obtained by projecting on \mathcal{B} the category of cocones from D to the elements of a *solution set* Σ . On the right-hand side of (1), there are two rather special cases of these constructions. The numerator shows a formula which holds in Boolean algebras – perhaps better known in its logical form

$$\exists x. \delta(x) = \neg \forall x. \neg \delta(x). \tag{2}$$

The denominator represents Paré’s Theorem [28; 1, 5.1], i.e. the calculation of a colimit using it. The setup for this formula is summarized on the following picture, where \mathcal{S} is a topos and \wp its power-set functor. $\mathcal{S}^{\mathbb{C}}$ is the category of \mathbb{C} -diagrams in \mathcal{S} and the “diagonal” functor $\Delta: \mathcal{S} \rightarrow \mathcal{S}^{\mathbb{C}}$ takes each object A of \mathcal{S} to the constant diagram on it.

$$\begin{array}{ccc}
 \mathcal{S}^{op} & \xrightarrow{\Delta^{op}} & (\mathcal{S}^{\mathbb{C}^{op}})^{op} \\
 \uparrow \wp^{op} & \xleftarrow{\lim^{op}} & \cong \\
 & & (\mathcal{S}^{op})^{\mathbb{C}} \\
 \downarrow \wp & & \downarrow \wp \circ (-) \\
 \mathcal{S} & \xrightarrow{\Delta} & \mathcal{S}^{\mathbb{C}} \\
 & \xleftarrow{\lim} &
 \end{array}
 \tag{3}$$

While the General Adjoint Functor Theorem is basically an existential statement, which only allows actual calculation when a particular solution set is provided, the special case captured in Paré’s Theorem is completely effective – probably the best way to calculate the coequalizers even in the category of sets. We shall lift these two theorems to the fibrational setting in, respectively, Sections 2–3 and 4–5 of this paper.

The common prerequisite for both these parts will be a “logical” formulation of the Adjoint Functor Theorem [34] – which is, conceptually, the common denominator for both sides of (1). It is formulated in Theorem 1.2. To motivate it, let us first point out that (2) is an instance ($\xi = \perp$) of the well-known formula of second-order logic:

$$\exists x. \delta(x) = \forall \xi (\forall x. \delta(x) \rightarrow \xi) \rightarrow \xi, \tag{4}$$

where ξ varies over all propositions. This formula suggests an intermediary form of proportions (1), applicable in the setting of complete Heyting algebras and Cartesian closed categories,

$$\begin{aligned} \vee D &= \bigwedge_{\xi} \bigwedge_d (d \rightarrow \xi) \rightarrow \xi \\ \dots &= \frac{\lim_{\rightarrow} D \hookrightarrow \prod_{X \in \Sigma} [\lim_{\leftarrow} [D, X], X]}{\dots} = \dots \end{aligned} \tag{1'}$$

The variable ξ runs over the whole algebra, while d takes its values in the subset D . The point is now that there is a way to state this formula in every complete category \mathcal{B} . The setup echoes diagram (3). First consider the functors

$$[-, -]: \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow \underline{\text{Set}} : \langle A, B \rangle \mapsto \mathcal{B}(A, B), \tag{5}$$

$$[-, -]: \underline{\text{Set}}^{\text{op}} \times \mathcal{B} \rightarrow \mathcal{B} : \langle J, B \rangle \mapsto \prod_j B. \tag{6}$$

The former is just the hom-set functor in a nonstandard notation; the latter produces the mixed exponents. Given a set Σ , define the functors

$$P: (\underline{\text{Set}}/\Sigma)^{\text{op}} \rightarrow \mathcal{B} : f \mapsto \prod_{X \in \Sigma} [f^{-1}\{X\}, X], \tag{7}$$

$$P_i: \mathcal{B} \rightarrow (\underline{\text{Set}}/\Sigma)^{\text{op}} : B \mapsto \left(\coprod_{X \in \Sigma} [B, X] \rightarrow \Sigma \right), \tag{8}$$

where \coprod denotes the disjoint union. They are adjoint and induce square (9). If Σ is a solution set of a diagram $D \in \mathcal{B}^{\mathcal{C}}$, then sending this diagram around (9) yields a weak colimit.

$$\begin{array}{ccc} (\underline{\text{Set}}/\Sigma)^{\text{op}} & \xrightarrow{\Delta^{\text{op}}} & ((\underline{\text{Set}}/\Sigma)^{\mathcal{C}})^{\text{op}} \\ & \xleftarrow{\lim^{\text{op}}} & \cong \\ & & ((\underline{\text{Set}}/\Sigma)^{\mathcal{C}})^{\text{op}} \\ \begin{array}{c} \uparrow P_i \\ \mathcal{B} \end{array} & \xrightarrow{\Delta} & \begin{array}{c} \uparrow P_i \circ (-) \\ \mathcal{B}^{\mathcal{C}} \end{array} \\ \begin{array}{c} \downarrow P \\ \mathcal{B} \end{array} & \xrightarrow{\Delta} & \begin{array}{c} \downarrow P \circ (-) \\ \mathcal{B}^{\mathcal{C}} \end{array} \end{array} \tag{9}$$

1.2. Theorem. *If Σ is a solution set for a small diagram D in a complete category \mathcal{B} , the object*

$$W := \prod_{X \in \Sigma} \left[\lim_{\leftarrow} [D, X], X \right] \tag{10}$$

is a weak colimit of D . The strong colimit Q of D can be obtained as the equalizer

$$Q \longrightarrow W \begin{array}{c} \xrightarrow{\langle id_W \rangle_{\text{rev}}} \\ \xrightarrow{\langle v \rangle_{\text{rev}}} \end{array} [V, W], \tag{11}$$

where V is the set of all endomorphisms of W which equalize the weakly initial cocone $\omega: D \rightarrow W$.

$$V := \{v \in [W, W] \mid v \circ \omega = \omega\}. \tag{12}$$

To remain faithful to the tradition of presenting the adjoint functor theorems, we leave the proof of this theorem and of the next one as an exercise for the reader. The details are in [34].

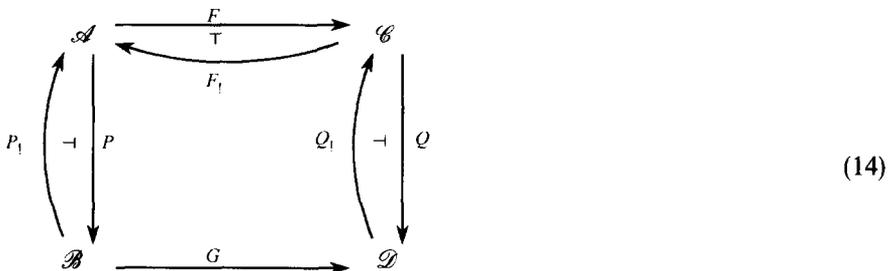
Slightly modified, the same idea yields an analogous formula for the General Adjoint Functor Theorem.

1.3. Theorem. *Let \mathcal{B} be a complete category and $G: \mathcal{B} \rightarrow \mathcal{D}$ a limit preserving functor. If Σ is a solution set for the object $D \in \mathcal{D}$, then*

$$W := \prod_{X \in \Sigma} \left[[D, GX], X \right] \tag{13}$$

*is weakly universal for G at D . The strongly universal object $G_*D := Q$ is obtained as in Theorem 1.2. If every object of \mathcal{D} has a solution set, then we can construct a functor G_* , left adjoint to G .*

1.4. Adjoint squares. A formula similar to (13) actually appears in the standard proof of the Special Adjoint Functor Theorem [9, 3.M; 26, V.8; 1, exercise SAFT]. The General Adjoint Functor Theorem is usually different. The above version points to a setting in which these and similar theorems can be effectively presented: that of *adjoint squares*.



An adjoint square consists of functors P, Q, F and G as above, such that there is a natural isomorphism $QF \simeq GP$. Furthermore, P and Q are assumed to have left adjoints $P_!$ and $Q_!$; alternatively, we sometimes assume right adjoints P_* and Q_* . This square is thus a morphism $\langle F, G \rangle: P \rightarrow Q$ in the category of functors which have an adjoint. The adjoints are considered as properties rather than data, and it is not required that they commute with F and G .

In an adjoint square, one typically asks: *Under which conditions does an adjoint of F induce an adjoint of G ?* Theorem 1.2 provides an answer to such a question: a weak colimit is derived from a solution set by going around adjoint square (9). In the rest of this section, we shall present three constructions where adjoint squares yield solution sets. The first one is slightly more general than the Special Adjoint Functor Theorem; the second one includes two *Butler's theorems* [38; 1, 3.7]; the third one is easy. To emphasize the parallelism and avoid repetitions, we have chosen to formulate them in a “polyphonic” way.

1.5. Propositions. *Suppose that a functor $P: \mathcal{A} \rightarrow \mathcal{B}$ has a left adjoint $P_!$ and that*

$$\mathcal{B} \left\{ \begin{array}{l} 1. \text{ is complete and well-powered} \\ 2. \text{ has equalizers} \\ 3. \text{ (no assumptions)} \end{array} \right\}.$$

Then for the statements

$$(a_{1,2,3}) \text{ The unit } \eta: id \rightarrow PP_! \text{ is componentwise } \left\{ \begin{array}{l} 1. \text{ monic} \\ 2. \text{ regular monic} \\ 3. \text{ iso} \end{array} \right\}.$$

(b_{1,2,3}) *For every $Q_! \dashv Q: \mathcal{C} \rightarrow \mathcal{D}$ and every morphism $\langle F, G \rangle: P \rightarrow Q$,*

$$\text{if } F \text{ has a left adjoint } F_! \text{ and } G \left\{ \begin{array}{l} 1. \text{ preserves limits} \\ 2. \text{ preserves equalizers} \\ 3. \text{ (no assumptions)} \end{array} \right\},$$

then G has a left adjoint $G_!$,

$$\text{with } \varphi: G_! \rightarrow PF_!Q_!, \text{ which is componentwise } \left\{ \begin{array}{l} 1. \text{ monic} \\ 2. \text{ regular monic} \\ 3. \text{ iso} \end{array} \right\}.$$

(c_{2,3}) *For every $Q_! \dashv Q: \mathcal{C} \rightarrow \mathcal{D}$ and every morphism $\langle F, G \rangle: P_! \rightarrow Q_!$,*

if F has a right adjoint F_ ,*

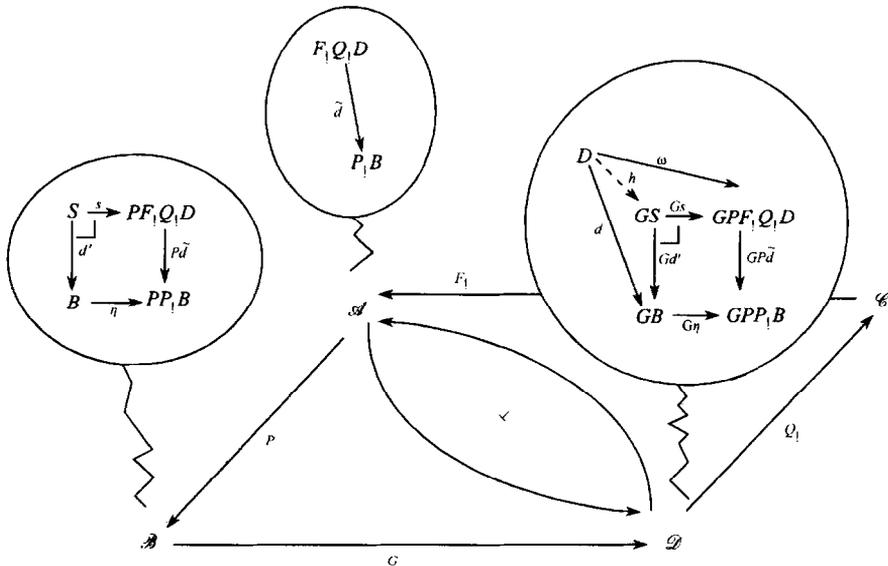
then G has a right adjoint G_ ,*

$$\text{with } \varphi: G_* \rightarrow PF_*Q_!, \text{ which is componentwise } \left\{ \begin{array}{l} 2. \text{ regular monic} \\ 3. \text{ iso} \end{array} \right\}.$$

the following equivalences hold:

- (1) $(a_1) \Leftrightarrow (b_1)$,
- (2) $(a_2) \Leftrightarrow (b_2) \Leftrightarrow (c_2)$,
- (3) $(a_3) \Leftrightarrow (b_3) \Leftrightarrow (c_3)$.

Proof. $(a_1) \Rightarrow (b_1)$ We first show that the subobjects of PF_1Q_1D form a solution set for $D \in \mathcal{D}$ with respect to $G: \mathcal{B} \rightarrow \mathcal{D}$. In other words, for every arrow $d: D \rightarrow GB$ in \mathcal{D} there is a subobject $S \rightarrow PF_1Q_1D$ with $d': S \rightarrow B$ in \mathcal{B} , such that $d = Gd' \circ h$ for some $h: D \rightarrow GS$ in \mathcal{D} , as shown in the following diagram.



(15)

The adjunction $F_1Q_1 \dashv QF$, with $QF \simeq GP$, induces $F_1Q_1 \dashv GP$. Transposed along this latter adjunction, the arrow $G\eta_B \circ d: D \rightarrow GB \rightarrow GPP_1B$ yields $\tilde{d}: F_1Q_1D \rightarrow P_1B$, such that

$$G\eta_B \circ d = GP\tilde{d} \circ \omega_D. \tag{16}$$

Here is $\eta: id \rightarrow PP_1$ the unit of $P_1 \dashv P$, while $\omega: id \rightarrow GPF_1Q_1$ is the unit of $F_1Q_1 \dashv GP$. Back in \mathcal{B} , the arrows $d': S \rightarrow B$ and $s: S \rightarrow PF_1Q_1D$ are obtained in a pullback of $P\tilde{d}$ and η_B . The functor G preserves this pullback and (15) induces $h: D \rightarrow GS$, with $d = Gd' \circ h$ and $\omega_D = Gs \circ h$. The subobjects of PF_1Q_1D thus form a solution set Σ for D . Theorem 1.3 now yields a left adjoint G_1 of G . (That construction is explicit, so this one is.)

It remains to be proved that G_1 will be included in PF_1Q_1 . If we put the unit $\gamma_D: D \rightarrow GG_1D$ of $G_1 \dashv G$ for d , (16) yields

$$G\eta_{G_1D} \circ \gamma_D = GP\tilde{\gamma}_D \circ \omega_D = GP\tilde{\gamma}_D \circ G(\omega_D) \circ \gamma_D, \tag{17}$$

where $\omega : G_1 \rightarrow PF_1Q_1$ is the transpose of $\omega : id \rightarrow GPF_1Q_1$. The two sides of (17) are the transposes of η_{G_1D} and of $P\tilde{\gamma}_D \circ \omega_D$, respectively. Since η is monic, $\omega : G_1 \rightarrow PF_1Q_1$ is monic too.

Implications $(a_2) \Rightarrow (b_2)$ and $(a_2) \Rightarrow (c_2)$ are dual to Butler’s theorems [1, 3.7.3]. In fact, the latter is just slightly stronger than a result from Lawvere’s thesis. The rest boils down to a sequence of routine exercises. \square

1.6. Remarks. Conditions $(a_{1,2,3})$ can be equivalently reformulated as

$$(a'_{1,2,3}) P_1 \text{ is } \left\{ \begin{array}{l} 1. \text{ faithful} \\ 2. \text{ of descent type} \\ 3. \text{ fully faithful} \end{array} \right\}.$$

Equivalences $(a_1) \Leftrightarrow (a'_1)$ and $(a_3) \Leftrightarrow (a'_3)$ are standard category theory; $(a_2) \Leftrightarrow (a'_2)$ is one of Beck’s theorems [1, 3.3, Theorem 9]. Note further that

- (1) P_1 is faithful iff the image of P cogenerates \mathcal{B} (i.e., the functors represented by the objects in the form PA are jointly faithful);
- (2) P_1 is of descent type iff the closure of the image of P under equalizers covers all of \mathcal{B} (i.e., every object is regularly included in some PA).

1.7. Examples. (A) Let \mathcal{A} be a closed symmetric monoidal category (or just closed, with coherent extranatural symmetry $\zeta_{AB}^C : [A, [B, C]] \rightarrow [B, [A, C]]$). Any object $\Omega \in \mathcal{A}$ induces a functor $P : \mathcal{A}^{op} \rightarrow \mathcal{A}$, defined $PA := [A, \Omega]$. This functor is self-adjoint, in the sense that $P^{op} \dashv P$. The situation from 1.5(3) arises when Ω is a dualizing object, i.e. when P is full and faithful – hence an equivalence. The category \mathcal{A} is self-dual, and the colimits can be constructed as limits and transferred along the duality P . This is the case of Barr’s **-autonomous* categories.

The situation from 1.5(2) arises when Ω is a *regular cogenerator* with respect to the closed structure. This means that the induced functor P is of descent type. The colimits of \mathcal{A} can be constructed as limits of PP^{op} -algebras and transferred back along the comparison functor, which is full and faithful. This situation is well-known from topos theory, where the power-set functor P is not only of descent type, but even monadic. Proposition 1.5(2) shows that it need not be: for the most important constructions, it is irrelevant whether it reflects isomorphisms or not. This can be understood as a mild generalization of Paré’s Theorem.

Proposition 1.5(1) extends this theorem to the situation when the “power-set” functor P is only faithful. This occurs, for instance, when \mathcal{A} is a complete and well-powered quasitopos (as most quasitoposes are). Sections 4 and 5 provide examples of a different kind.

- (B) Let \mathcal{B} be a complete category, with a distinguished object K – which induces

$$R : \underline{Set}^{op} \rightarrow \mathcal{B} : I \mapsto [I, K], \tag{18}$$

$$R_! : \mathcal{B} \rightarrow \underline{Set}^{op} : B \mapsto [B, K]. \tag{19}$$

R is faithful iff K is a cogenerator (in the external sense). In this case, Proposition 1.5(1) boils down to the Special Adjoint Functor Theorem. If \mathcal{B} is a variety and if K is an injective cogenerator, then $R_!^{\text{op}}$ is monadic, and 1.5(2) can be applied. The colimits of \mathcal{B} can be calculated as the limits of $(R_!R)^{\text{op}}$ -algebras on sets. Proposition 1.5(3) now corresponds to the situation when K is a dualizing object externally. The simplest example: \mathcal{B} is the category of complete atomic Boolean algebras and K is the lattice with two elements. The Stone duality is obtained when \mathcal{B} is extended to all Boolean algebras and Set to Stone spaces. Further modifications of this setup produce many other dualities.

2. In small categories

In the rest of the paper, all categories will be fibred (or indexed). For more about this setting, see [3, 13–15, 29].¹

2.1. Theorem. *Let \mathcal{S} be a finitely complete category and \mathbb{B} an \mathcal{S} -small complete category. Then \mathbb{B} is cocomplete if either*

- (A) \mathbb{B} is Cartesian closed; or if
- (B) \mathcal{S} is locally Cartesian closed.

Let us first explain what all this means. By definition [2], a fibred category is \mathcal{S} -small if it is equivalent to the externalization of an internal category in \mathcal{S} . Recall that an internal category \mathbb{B} in \mathcal{S} is a diagram

$$\begin{array}{ccc}
 & \xrightarrow{\partial_0} & \\
 B_2 & \xrightarrow{\mu} B_1 & \xleftarrow{\eta} B \\
 & \xleftarrow{\partial_1} &
 \end{array} \tag{20}$$

The idea is that B represents the set of objects, B_1 the set of arrows, and that ∂_0 and ∂_1 are the domain and the codomain operations. The arrow μ represents the composition, while η assigns to each object the identity on it. For more details, see [22, ch. 2].

Ordinary category theory is about categories fibred over Set. Namely, with every category \mathcal{B} , we are always given the categories \mathcal{B}^I of I -indexed families, where I is an arbitrary set. Together with the reindexing functors, these categories constitute a fibration over Set. In a similar way, every category \mathbb{B} internal in \mathcal{S} , induces a fibred category: its externalization \mathcal{S}/\mathbb{B} . The objects of \mathcal{S}/\mathbb{B} are the arrows to the object of objects B . The hom-set between $x: I \rightarrow B$ and $y: J \rightarrow B$ will be

$$\mathcal{S}/\mathbb{B}(x, y) := \{ \langle f, \varphi \rangle \mid f \in \mathcal{S}(I, J), \varphi \in \mathcal{S}(I, B_1), \langle \partial_0, \partial_1 \rangle \circ \varphi = \langle x, y \circ f \rangle \}. \tag{21}$$

¹ Following [30], we shall try to convey the basic ideas, which should enable the reader to understand the statements. To understand proofs, (s)he will probably need more. The differences between indexed and fibred categories are not essential for the constructions here. The term “fibration” usually denotes the functor by which a category is fibred.

Projecting $x: I \rightarrow B$ to I and $\langle f, \varphi \rangle$ to f yields the fibration $\nabla\mathbb{B}: \mathcal{S}/\mathbb{B} \rightarrow \mathcal{S}$. The “ I -indexed families” form the *fibres* over I – i.e. the subcategory \mathbb{B}_I of \mathcal{S}/\mathbb{B} consisting of the objects and the arrows which the functor $\nabla\mathbb{B}$ maps on I and its identity. The inverse image (i.e., reindexing) functors for the fibration $\nabla\mathbb{B}$ are

$$f^*: \mathbb{B}_J \rightarrow \mathbb{B}_I: y \mapsto y \circ f.$$

If a fibration $E: \mathcal{E} \rightarrow \mathcal{S}$ is equivalent with $\nabla\mathbb{B}: \mathcal{S}/\mathbb{B} \rightarrow \mathcal{S}$, the object $\xi \in \mathcal{E}_B$, which corresponds to $id \in \mathbb{B}_B$, is *generic*. Namely, if $Y \in \mathcal{E}_J$ corresponds to $y: J \rightarrow B$, then $Y = y^*\xi$. We say that y *classifies* Y . There is also a generic arrow $\gamma \in \mathcal{E}_{B_1}(\partial_0^*\xi, \partial_1^*\xi)$ – which corresponds along the equivalence to the arrow $\langle id_{B_1}, id_{B_1} \rangle \in \mathcal{S}/\mathbb{B}(\partial_0, \partial_1)$. We shall abuse notation and denote by $\nabla\mathbb{B}$ any fibration equivalent to the externalization of \mathbb{B} .

Now we want to explain the notion of (co)completeness. In general, all finitary operations on fibred categories are *fibrewise*: they are defined in each fibre and required to be stable under the inverse images. For instance, a small fibration \mathbb{B} is said to be Cartesian closed if each fibre \mathbb{B}_I has the finite products and the exponents, and the inverse image functors $f^*: \mathbb{B}_J \rightarrow \mathbb{B}_I$ preserve them. This turns out to be equivalent to saying that there are internal operations on the internal category \mathbb{B} , representing the finite products and the exponents. (Namely, the externalisation ∇ is a full and faithful 2-functor from the category of internal categories in \mathcal{S} to the category of fibrations over \mathcal{S} . This is the Yoneda embedding [30, III.1].)

The infinitary operations are dealt with in a different manner. In an ordinary category \mathcal{B} , the products of I -indexed families can be presented as a right adjoint functor to the diagonal $\mathcal{B} \rightarrow \mathcal{B}^I$, the reindexing along the function $I \rightarrow 1$. The I -indexed coproducts are left adjoint to this diagonal. It is easy to see that \mathcal{B} has all small (co)products if and only if the reindexing functors over all functions $f: I \rightarrow J$ have right (resp. left) adjoints. Accordingly, for categories fibred over \mathcal{S} , the small products and coproducts are defined, respectively, as the right and the left adjoints to the reindexing. The small products in an \mathcal{S} -small category \mathbb{B} over the arrow $f: I \rightarrow J$ in \mathcal{S} are thus given by a functor $f_*: \mathbb{B}_I \rightarrow \mathbb{B}_J$, right adjoint to f^* . The small coproducts are given by a left adjoint $f_!: \mathbb{B}_I \rightarrow \mathbb{B}_J$. There is also a proviso of stability with respect to the reindexing, expressed by the *Beck–Chevalley conditions* [24] (also explained in [33]). Contrary to the situation with finitary operations, these *fibrational* (co)products are essentially stronger than the corresponding internal notions [41, 21]. A fibred category is said to be *(co)complete* if it has the fibrational (co)products and the finite (co)limits fibrewise.

Note, finally, that the locally Cartesian closed structure of \mathcal{S} is a fibrewise structure with respect to the *basic* fibration $\nabla\mathcal{S} = Cod: \mathcal{S}/\mathcal{S} \rightarrow \mathcal{S}$ – which can be understood as the “externalization” of \mathcal{S} itself. The fibred versions of functors (5) and (6), which will be needed in the proof of the above theorem, are also formulated using $\nabla\mathcal{S}$. An \mathcal{S} -small fibred category is, of course, a fortiori \mathcal{S} -locally small. A hom-set representant $\lfloor X, Y \rfloor: \mathbb{B}^J(X, Y) \rightarrow J$ for the objects $X, Y \in \mathbb{B}_J$, can be obtained by pulling back the arrow $\langle \partial_0, \partial_1 \rangle: B_1 \rightarrow B \times B$ along $\langle x, y \rangle: J \rightarrow B \times B$, where

$X = x^*\xi$ and $Y = y^*\xi$. Thus, $\mathbb{B}^J(X, Y)$ is an object of \mathcal{S} , while $\mathbb{B}_J(X, Y)$ denotes the actual hom-set of the fibre $\mathbb{B}_J = (\mathcal{S}/\mathbb{B})_J$. The former is a representant of the latter – in the sense that there is a correspondence

$$\mathcal{S}/J(f, \lfloor X, Y \rfloor) \simeq \mathbb{B}_I(f^*X, f^*Y), \tag{22}$$

natural in $f \in \mathcal{S}(I, J)$ and $X, Y \in |\mathbb{B}_J|$. This representation is stable:

$$f^*\lfloor X, Y \rfloor \simeq \lfloor f^*X, f^*Y \rfloor \tag{23}$$

and yields a Cartesian (i.e., inverse image preserving) functor

$$\lfloor -, - \rfloor : \mathcal{S}/(\mathbb{B}^{\text{op}} \times \mathbb{B}) \rightarrow \mathcal{S}/\mathcal{S}. \tag{24}$$

This is the fibrational version of (5). The fibrational version of mixed exponents (6) will be defined using the small products of \mathbb{B} :

$$\lceil -, - \rceil_J : (\mathcal{S}/J)^{\text{op}} \times \mathbb{B}_J \rightarrow \mathbb{B}_J : \langle f, Z \rangle \mapsto \lceil f, Z \rceil := f_*f^*Z. \tag{25}$$

In the proof below, we shall show how to define the arrow part of this functor. The fibrational version of (6), the Cartesian functor

$$\lceil -, - \rceil : \mathcal{S}/(\mathcal{S}^{\text{op}} \times \mathbb{B}) \rightarrow \mathcal{S}/\mathbb{B}. \tag{26}$$

is obtained by letting J in (25) run over \mathcal{S} . The fibred category $\mathcal{S}/(\mathcal{S}^{\text{op}} \times \mathbb{B})$ can be obtained as the pullback of the fibrations $\nabla\mathbb{B} : \mathcal{S}/\mathbb{B} \rightarrow \mathcal{S}$ and $\nabla\mathcal{S}^{\text{op}} : \mathcal{S}/\mathcal{S}^{\text{op}} \rightarrow \mathcal{S}$. In the latter, the arrows in the fibres of the basic fibration $\nabla\mathcal{S} : \mathcal{S}/\mathcal{S} \rightarrow \mathcal{S}$ are formally inverted [30, II.2.1]

Proof of the Theorem. The idea is to encode formulas (1') and (10), taking all the objects of \mathbb{B} as a solution set. The generic object ξ of \mathbb{B} is used as a variable running over them.

(A) To construct a direct image functor $g_! \dashv g^*$ over an arbitrary $g \in \mathcal{S}(I, J)$, we shall apply Proposition 1.5(2) to the scheme

$$\begin{array}{ccc}
 \mathbb{B}_{J \times B}^{\text{op}} & \xrightarrow{\quad ((g \times id)^*)^{\text{op}} \quad} & \mathbb{B}_{I \times B}^{\text{op}} \\
 \uparrow P_! \dashv P & \xleftarrow{\quad (g \times id)_*^{\text{op}} \quad} & \uparrow Q_! \dashv Q \\
 \mathbb{B}_J & \xrightarrow{\quad g^* \quad} & \mathbb{B}_I
 \end{array} \tag{27}$$

with the functors

$$PX := \pi_*[X, \rho^*\xi],$$

$$P_!Y := [\pi^*Y, \rho^*\xi],$$

where $\pi: J \times B \rightarrow J$ and $\rho: J \times B \rightarrow B$ are projections. It is easy to see that the conditions for 1.5(2) are fulfilled. In Lemma 2.2(A), we shall show that the unit $\eta: id \rightarrow PP_i$ is a split monic. It is thus regular, and 1.5(a₂) is true. Applying 1.5(b₂), one now constructs $g_i \sqsubset P \circ (g \times id)_*^{op} \circ Q_i$. It satisfies the Beck–Chevalley condition because $(g \times id)_*$ does. So g_i presents the small coproducts.

The finite colimits in \mathbb{B}_I can be obtained by sending diagrams around the square

$$\begin{array}{ccc}
 \mathbb{B}_{I \times B}^{op} & \xrightarrow{\Delta^{op}} & (\mathbb{B}_{I \times B}^{c^{op}})^{op} \\
 \uparrow P_i \dashv P & \swarrow \lim^{op} & \cong \\
 & & (\mathbb{B}_{I \times B}^{op})^c \\
 & & \downarrow P_i \circ (-) \dashv P \circ (-) \\
 \mathbb{B}_I & \xrightarrow{\Delta} & \mathbb{B}_I^c
 \end{array} \tag{28}$$

(B) When \mathcal{S} is locally Cartesian closed, instead of (27), we form (29).

$$\begin{array}{ccc}
 (\mathcal{S}/J \times B)^{op} & \xrightarrow{((g \times id)_*)^{op}} & (\mathcal{S}/I \times B)^{op} \\
 \uparrow R_i \dashv R & \swarrow (g \times id)_*^{op} & \uparrow S_i \dashv S \\
 \mathbb{B}_I & \xrightarrow{g^*} & \mathbb{B}_I
 \end{array} \tag{29}$$

The adjunctions are defined using (24) and (26):

$$R_x := \pi_* \lceil x, \rho^* \xi \rceil,$$

$$R_i Y := \lceil \pi^* Y, \rho^* \xi \rceil.$$

The image Rh of $h \in \mathcal{S}/J \times B(z, x)$ is derived from the unit ω of the adjunction $h^* \dashv h_*$:

$$\frac{\omega: x^* \rho^* \xi \rightarrow h_* h^* x^* \rho^* \xi \text{ in } \mathbb{B}_{Dom(x)}}{R_h := \pi_* x_* \omega: \pi_* x_* x^* \rho^* \xi \rightarrow \pi_* x_* h_* h^* x^* \rho^* \xi \simeq \pi_* z_* z^* \rho^* \xi \text{ in } \mathbb{B}_J}$$

It is straightforward to prove that R_i is left adjoint to R . Lemma 2.2(B) tells that the unit $\eta: id \rightarrow RR_i$ is split monic again, so that 1.6(a₂) holds. The coproduct functor g_i is again constructed using 1.6(b₂). This time it is less straightforward to show that $F = ((g \times id)_*)^{op}$ and $G = g^*$ form an adjoint square with R and Q . We apply the

Beck–Chevalley condition.

(30)

$$\begin{aligned}
 GRx &= g^* \pi_* \lceil x, \rho^* \xi \rceil = g^* \pi_* x_* x^* \rho^* \xi \simeq \pi'_* ((g \times id)^* x)_* q^* x^* \rho^* \xi \\
 &\simeq \pi'_* ((g \times id)^* x)_* ((g \times id)^* x)^* ((g \times id)^* \rho^* \xi) \\
 &\simeq \pi'_* ((g \times id)^* x)_* ((g \times id)^* x)^* \rho^* \xi \\
 &= \pi'_* \lceil (g \times id)^* x, \rho^* \xi \rceil = SFx
 \end{aligned}$$

The finite colimits in \mathbb{B}_I are obtained from the finite limits in $\mathcal{S}/I \times B$. The reader is invited to draw a suitable version of diagram (28), which resembles (9). \square

2.2. Lemma. *The units of adjunctions (A) $P_1 \dashv P$ and (B) $R_1 \dashv R$ are split monics.*

Proof. (A) The assertion is proved by chasing diagram (31) appearing on the next page. The arrow y classifies Y .

$\omega: id \rightarrow \pi_* \pi^*$ and $\upsilon: \pi^* \pi_* \rightarrow id$ are the unit and the counit of the adjunction $\pi^* \dashv \pi_*$, while $\alpha_Z: Z \rightarrow [[Z, S], S]$ is the usual transpose of the twisted evaluation. $\lceil id \rceil$ is obtained by composing the “name of the identity” $\top J \rightarrow [Y, Y]$ and the arrow $[[Y, Y], Y] \rightarrow \top J$, where $\top J$ is the terminal object of \mathcal{E}_J .

Isomorphisms I are induced by the fact that $\pi \circ \langle id, y \rangle = id$; II commutes by the definition of η , and III by the naturality of υ . Isomorphism IV follows from $Y = y^* \xi$, using the stability of the exponents. The rectangle with IV commutes because α is stable too, and V because

$$\begin{aligned}
 e_{[Y, Y]} \circ \langle \alpha_Y, \lceil id_Y \rceil \rangle &= e_{[Y, Y]} \circ (\alpha_Y \times id_{[Y, Y]}) \circ \langle id_Y, \lceil id_Y \rceil \rangle \\
 &= e_Y \circ \tau \circ \langle id_Y, \lceil id_Y \rceil \rangle = e_Y \circ \langle \lceil id_Y \rceil, id_Y \rangle = id_Y.
 \end{aligned}$$

$$\begin{array}{ccc}
 Y & \xrightarrow{\eta_Y} & \pi_*[[\pi^*Y, \rho^*\xi], \rho^*\xi] \\
 \downarrow \cong \text{I} & & \downarrow \cong \\
 \langle id, y \rangle^* \pi^* Y & \xrightarrow{\langle id, y \rangle^* \pi^* \eta_Y} & \pi_*[[\pi^*Y, \rho^*\xi], \rho^*\xi] \\
 \downarrow \text{II} & & \downarrow \text{I} \\
 \langle id, y \rangle^* \pi^* \omega & \xrightarrow{\langle id, y \rangle^* \pi^* \omega} & \langle id, y \rangle^* \pi^* \pi_*[[\pi^*Y, \rho^*\xi], \rho^*\xi] \\
 \downarrow \text{III} & & \downarrow \langle id, y \rangle^* \nu_{[\cdot, \cdot], \rho^*\xi} \\
 \langle id, y \rangle^* \pi^* \pi_* \pi^* Y & \xrightarrow{\langle id, y \rangle^* \pi^* \omega} & \langle id, y \rangle^* \pi^* \pi_*[[\pi^*Y, \rho^*\xi], \rho^*\xi] \\
 \downarrow \text{IV} & & \downarrow \text{IV} \\
 \langle id, y \rangle^* \pi^* Y & \xrightarrow{\langle id, y \rangle^* \omega} & \langle id, y \rangle^* [[\pi^*Y, \rho^*\xi], \rho^*\xi] \\
 \downarrow \cong \text{I} & & \downarrow \cong \\
 Y & \xrightarrow{\omega} & [[Y, Y], Y] \\
 \downarrow \text{V} & & \downarrow \langle id, id^\top \rangle \\
 Y & \xrightarrow{e} & [[Y, Y], Y] \times [Y, Y]
 \end{array} \tag{31}$$

The left-hand composite along (31) is the identity, since $\nu_{\pi^*} \circ \pi^* \omega = id_{\pi^*}$. The right-hand composite thus yields a left inverse for $\eta_Y: Y \rightarrow PP_1 Y$.

(B) The idea is the same as under (A), although some details are more subtle. However, the analogy of structures is not as smooth as our notation might suggest. Already the unit $\eta_Y: Y \rightarrow RR_1 Y$ is derived quite differently than before, using (22) instead of the closed structure.

$$\frac{
 \begin{array}{l}
 id: [\pi^*Y, \rho^*\xi] \rightarrow [\pi^*Y, \rho^*\xi] \text{ in } \mathcal{S}/J \times B \\
 q: [\pi^*Y, \rho^*\xi]^* \pi^* Y \rightarrow [\pi^*Y, \rho^*\xi]^* \rho^*\xi \text{ in } \mathbb{B}_H
 \end{array}
 }{
 \frac{
 \omega: \pi^* Y \rightarrow [\pi^*Y, \rho^*\xi]_* [\pi^*Y, \rho^*\xi]^* \rho^*\xi \text{ in } \mathbb{B}_{J \times B}
 }{
 \eta: Y \rightarrow \pi_* [[\pi^*Y, \rho^*\xi], \rho^*\xi] \text{ in } \mathbb{B}_B
 }
 }$$

$H \in \mathcal{S}$ is the domain of $[\pi^*Y, \rho^*\xi]$. The new version of diagram (31) is diagram (32) appearing on the next page.

I–III commute for similar reasons as in case (A). The “name of the identity” is now the arrow $\lceil id^\top \rceil \in \mathcal{S}/J(id_J, [Y, Y])$, corresponding by (22) to $id \in \mathbb{B}_J(Y, Y)$. Isomorphism IV thus follows from $[Y, Y] \circ \lceil id^\top \rceil = id_J$. Isomorphism V is derived as follows:

$$\begin{aligned}
 \langle id, y \rangle^* [[\pi^*Y, \rho^*\xi], \rho^*\xi] &= \langle id, y \rangle^* [\pi^*Y, \rho^*\xi]_* [\pi^*Y, \rho^*\xi]^* \rho^*\xi \\
 &\stackrel{(\#)}{\cong} [Y, Y]_* \nu^* [\pi^*Y, \rho^*\xi]^* \rho^*\xi \\
 &\cong [Y, Y]_* (\rho \circ [\pi^*Y, \rho^*\xi] \circ \nu)^* \xi
 \end{aligned}$$

$$\begin{array}{ccc}
 Y & \xrightarrow{\eta_Y} & \pi_*[\lfloor \pi^*Y, \rho^*\xi \rfloor, \rho^*\xi] \\
 \downarrow \cong \text{I} & \searrow \langle id, y \rangle^* \pi^* \eta_Y & \downarrow \cong \\
 \langle id, y \rangle^* \pi^* Y & \xrightarrow{\langle id, y \rangle^* \pi^* \omega} & \langle id, y \rangle^* \pi_* \pi^* Y \\
 \downarrow \text{II} & \searrow \langle id, y \rangle^* \pi^* \omega & \downarrow \langle id, y \rangle^* \\
 \langle id, y \rangle^* \pi^* \pi_* \pi^* Y & \xrightarrow{\langle id, y \rangle^* \pi^* \pi_* \omega} & \langle id, y \rangle^* \pi_* \pi^* \pi_* \pi^* Y \\
 \downarrow \text{III} & \searrow \langle id, y \rangle^* \pi^* \omega & \downarrow \langle id, y \rangle^* \\
 \langle id, y \rangle^* \pi^* Y & \xrightarrow{\langle id, y \rangle^* \omega} & \langle id, y \rangle^* \pi_* \pi^* Y \\
 \downarrow \cong \text{I} & \searrow \phi & \downarrow \cong \text{V} \\
 Y & \xrightarrow{\phi} & \llbracket Y, Y, Y \rrbracket \\
 \downarrow \cong \text{IV} & \searrow & \downarrow \cong \text{IV} \\
 \ulcorner id \urcorner^* \llbracket Y, Y \rrbracket^* Y & \xrightarrow{\ulcorner id \urcorner^* \llbracket Y, Y \rrbracket^*} & \ulcorner id \urcorner^* \llbracket Y, Y \rrbracket^* \llbracket Y, Y, Y \rrbracket \xrightarrow{\ulcorner id \urcorner^* \theta_{\llbracket Y, Y \rrbracket^*}} \ulcorner id \urcorner^* \llbracket Y, Y \rrbracket^* Y
 \end{array} \tag{32}$$

$$\begin{aligned}
 &= \llbracket Y, Y \rrbracket_* (\rho \circ \langle id, y \rangle \circ \llbracket Y, Y \rrbracket)^* \xi \\
 &= \llbracket Y, Y \rrbracket_* (y \circ \llbracket Y, Y \rrbracket)^* \xi \simeq \llbracket Y, Y \rrbracket_* \llbracket Y, Y \rrbracket^* y^* \xi \\
 &\simeq \llbracket Y, Y \rrbracket_* \llbracket Y, Y \rrbracket^* Y
 \end{aligned}$$

At step (#), we use the Beck–Chevalley condition over square (33), which is a pullback by (23).

$$\begin{array}{ccccc}
 & \xrightarrow{v} & A & & \\
 \lrcorner & & \downarrow \llbracket \pi^* Y, \rho^* \xi \rrbracket & & \\
 \llbracket Y, Y \rrbracket & & & & \\
 \downarrow & & & & \\
 J & \xrightarrow{\langle id, y \rangle} & J \times B & \xrightarrow{\rho} & B
 \end{array} \tag{33}$$

The natural transformations ω and v are the unit and the counit of the adjunction $\pi^* \dashv \pi_*$, while ϕ and θ are the unit and the counit of $\llbracket Y, Y \rrbracket^* \dashv \llbracket Y, Y \rrbracket_*$. Hence $v_{\pi^*} \circ \pi^* \omega = id_{\pi^*}$ and $\theta_{\llbracket Y, Y \rrbracket^*} \circ \llbracket Y, Y \rrbracket^* \phi = id_{\llbracket Y, Y \rrbracket^*}$. Going along the left-hand side and the bottom of (39) thus yields the identity. The right-hand side is a left inverse of $\eta_Y: Y \rightarrow \llbracket Y, Y \rrbracket_*$. \square

Repeating the step from 1.2 to 1.3, we can now easily prove the General Adjoint Functor Theorem. For brevity, we only formulate it for small complete categories, although the size restriction could easily be replaced by a solution set. Paré and Schumacher [29, Section IV] have proved such a version, but assuming a very strong

notion of solution set, with the boundaries extracted. (cf. [34, Section 4].) Moreover, the methods of their monograph essentially require that the base category \mathcal{S} is locally Cartesian closed – i.e., that it has *small hom-sets*, as the authors put it. At any rate, our Theorem 2.1(B) was derived as a consequence [29, V.1.1].

But what does it mean that a fibred category $E: \mathcal{E} \rightarrow \mathcal{S}$ has small hom-sets? We need this general concept below. For $J \in \mathcal{S}$ and $X, Y \in \mathcal{E}_J$ the set $HOM(X, Y)$ varies over \mathcal{S} by the functor

$$HOM(X, Y): (\mathcal{S}/J)^{op} \rightarrow \underline{Set}: f \mapsto \mathcal{E}_I(f^*X, f^*Y), \tag{34}$$

where I is the domain of f . The fibration E is *locally small* if all $HOM(X, Y)$ are representable [2, 11, 2.6.2–3]. A representant $\lfloor X, Y \rfloor \in \mathcal{S}/J$ thus provides a natural correspondence

$$\mathcal{S}/J(f, \lfloor X, Y \rfloor) \simeq \mathcal{E}_I(f^*X, f^*Y). \tag{22'}$$

2.3. Theorem. *Let \mathcal{S} be a finitely complete category, \mathcal{E} a locally small and \mathbb{B} a small complete category, both fibred over \mathcal{S} . A Cartesian functor $G: \mathcal{S}/\mathbb{B} \rightarrow \mathcal{E}$ has a left adjoint if and only if it preserves all limits (in fact, the small products and the fibrewise equalizers suffice).*

By definition, G is Cartesian if for every $f \in \mathcal{S}(I, J)$ holds $G_I f^* \simeq f^* G_J$, where $G_I: \mathbb{B}_I \rightarrow \mathcal{E}_I$ is a restriction of G . The preservation of the small products, on the other hand, means $G_J f_* \simeq f_* G_I$.

Proof. This time we use the adjoint square

$$\begin{array}{ccc}
 (\mathcal{S}/J \times B)^{op} & \xrightarrow{id} & (\mathcal{S}/J \times B)^{op} \\
 \left. \begin{array}{c} \uparrow R \\ \lrcorner \dashv \\ \downarrow R \end{array} \right\} & & \left. \begin{array}{c} \uparrow L \\ \lrcorner \dashv \\ \downarrow G_J R \end{array} \right\} \\
 \mathbb{B}_J & \xrightarrow{G_J} & \mathcal{E}_J
 \end{array} \tag{35}$$

with R as in the proof of 2.1(B) and

$$LY := \lfloor \pi^*Y, \rho^*G_B(\xi) \rfloor.$$

The fact that L is left adjoint to $G_J R$ follows from the preservation properties of G :

$$\begin{aligned}
 \mathcal{E}_J(Y, G_J \pi_* \lceil x, \rho^* \xi \rceil) &\simeq \mathcal{E}_{J \times B}(\pi^*Y, \lceil x, \rho^* G_B \xi \rceil) \simeq \mathcal{E}_A(x^* \pi^* Y, x^* \rho^* G_B \xi) \\
 &\simeq \mathcal{S}/J \times B(x, \lfloor \pi^*Y, \rho^* G_B \xi \rfloor).
 \end{aligned}$$

Lemma 2.2(B) now says that $R_1 \dashv R$ satisfies condition 1.6(a₂). Since G_J preserves equalizers, the premises of 1.6(b₂) are fulfilled, and we get $H_J \dashv G_J$. The Beck–Chevalley condition now induces coherent isomorphisms $H_I \circ f^* \simeq f^* \circ H_J$ for all $f \in \mathcal{S}(I, J)$. (Tedious but straightforward.) As J runs over \mathcal{S} , the functors H_J thus determine Cartesian functor $H \dashv G$. \square

2.4. Remark. The other way around, does the cocompleteness of \mathbb{B} imply the completeness? Does a colimit-preserving functor G necessarily have a right adjoint? The answers are simple and affirmative – due to the simple way in which the opposite categories of internal categories are formed: by interchanging the domain ∂_0 and the codomain ∂_1 .

Dual of Theorem 2.1. *In a finitely complete category \mathcal{S} , the cocompleteness of a small category \mathbb{B} implies its completeness – provided that either \mathcal{S} is locally Cartesian closed, or that \mathbb{B}^{op} is Cartesian closed.*

Dual of Theorem 2.3. *A Cartesian functor $G: \mathcal{S}/\mathbb{B} \rightarrow \mathcal{E}$ has a right adjoint if and only if it preserves colimits – assuming that \mathcal{S} is a finitely complete category, while the fibred categories \mathcal{E} and \mathbb{B} are, respectively, locally small and small cocomplete.*

2.5. Example. For the readers who might not have seen it, we briefly describe the small complete category of partial equivalence relations, or *pers* for short. Other examples can be obtained from this one as (co)algebras, especially as actions, or as reflective subcategories [10]. The main idea is to extend the notion of *realizability* on sets: to encode them by natural numbers and to trace the functions by partial recursive functions on the codes. An *encoded set* (or *assembly* [5], or *ω -set* [25]) is a function $K: |K| \rightarrow \Sigma^*$, where $|K|$ is any set, while Σ^* is the set of the nonempty sets of natural numbers. An arrow $f: K \rightarrow L$ between two encoded sets is just a function $f: |K| \rightarrow |L|$, but *effectively encodable*: there must exist a natural number $n \in \mathbb{N}$, encoding a partial function $n': \mathbb{N} \rightrightarrows \mathbb{N}$, which is defined on $K(x)$ for every $x \in |K|$, and maps it into $L(f(x))$. This is summarized in diagram (36), where n'' denotes the partial direct image of n' , defined on $X \in \Sigma^*$ if and only if n' is defined on all of X

$$\begin{array}{ccc}
 |K| & \xrightarrow{f} & |L| \\
 \downarrow K & \subseteq & \downarrow L \\
 \Sigma^* & \xrightarrow{\circlearrowleft n''} & \Sigma^*
 \end{array} \tag{36}$$

Encoded sets and functions form a category \mathcal{S} , equivalent to the category of $\dashv\vdash$ -separated objects in the Effective Topos [18, Proposition 6.1., 5, Proposition 9]. This is a quasi-topos.

Pers form in \mathcal{S} an internal category \mathbb{B} . The object of objects is the set

$$|B| = \{\alpha \subseteq \Sigma^* \mid \forall xy \in \alpha. x \cap y = \emptyset\}; \text{ encoded by}$$

$$B(\alpha) = \mathbb{N}. \tag{37}$$

The inclusion $\alpha \subseteq \Sigma^*$ can be regarded as the natural encoding for each $\alpha \in |B|$. In this way, the objects α of \mathbb{B} appear as objects of \mathcal{S} . This allows us to define the object of arrows B_1 of \mathbb{B} in \mathcal{S} :

$$|B_1| = \{\langle \alpha, \beta, f \rangle \mid \alpha, \beta \in |B|; f \in \mathcal{S}(\alpha, \beta)\},$$

$$B_1\langle \alpha, \beta, f \rangle = \{n \in \mathbb{N} \mid n'' \circ \alpha \subseteq \beta \circ f\}. \tag{38}$$

The domain arrow ∂_0 and the codomain ∂_1 are the obvious projections from B_1 to B . \mathbb{B} is complete and cocomplete: the fibration $\mathbb{V}\mathbb{B}: \mathcal{S}/\mathbb{B} \rightarrow \mathcal{S}$ is fibrationally complete and cocomplete. Observe first that every object $\gamma: I \rightarrow B$ of \mathcal{S}/\mathbb{B} can be viewed as an ordinary indexed family $\langle \gamma(i) \rangle_{i \in |I|}$ of the objects of \mathbb{B} . Namely, since the encoding of B is trivial, all functions from $|I|$ to $|B|$ are \mathcal{S} -morphisms from I to B . Now the coproducts and the products of $\langle \gamma(i) \rangle_{i \in |I|}$ along $f \in \mathcal{S}(I, J)$ can be defined, respectively, as indexed families of pers

$$f_! \langle \gamma(i) \rangle_{i \in |I|} = \langle !\gamma(j) \rangle_{j \in |J|} \tag{39}$$

and

$$f_* \langle \gamma(i) \rangle_{i \in |I|} = \langle *\gamma(j) \rangle_{j \in |J|}, \tag{40}$$

where $!\gamma(j)$ is a quotient of the set-theoretical coproduct $\coprod_{f(i)=j} \gamma(i)$; while $*\gamma(j)$ arises as a subset of the product $\prod_{f(i)=j} \gamma(i)$. More precisely, each element z of $!\gamma(j)$ is the union of one equivalence class of elements of $\coprod_{f(i)=j} \gamma(i)$, modulo the transitive closure of the relation

$$x \sim y : \Leftrightarrow x \cap y \neq \emptyset. \tag{41}$$

On the other hand, each element z of $*\gamma(j)$ is the intersection of all the components z_i of some $\langle z_i \rangle \in \prod_{f(i)=j} \gamma(i)$, such that this intersection is nonempty. The reader may wish to check that (39) and (40) indeed define the coproducts and the products and that they can be derived from each other.

3. In small full subcategories

3.1. Examples, ideas. By definition (38), the morphisms of pers coincide with the encoded set morphisms between them: intuitively, they form a full subcategory. This fact can be used for a more economic internal description of the former category in the latter, omitting the object of arrows, the composition etc. Instead, we make the disjoint union of all pers,

$$|\mathcal{E}| = \{\langle \alpha, x \rangle \mid x \in \alpha \in |B|\} \text{ and encode it by}$$

$$\mathcal{E}\langle \alpha, x \rangle = x. \tag{42}$$

This encoded set is, of course, projected on B from (38) by $\iota \langle \alpha, x \rangle = \alpha$. The morphism $\iota: \mathcal{E} \rightarrow B$ in the category \mathcal{S} of encoded sets represents pers as an indexed family: each per α can be obtained as an object of \mathcal{S} by pulling back ι along the arrow $1 \rightarrow B$, which picks out $\alpha \in B$.

A full subcategory in any \mathcal{S} is determined by saying what its objects are. An internal full subcategory \mathcal{L} is thus an internally indexed family of objects $\iota: \mathcal{E} \rightarrow B$. The codomain B is the “set of indices”, while \mathcal{E} is the “disjoint union” of the objects of \mathfrak{B} . Every set of sets $\{X_b\}_{b \in B}$ can be presented in this way; and any full subcategory \mathfrak{B} of \underline{Set} is spanned by such a set. Another instructive example is the truth arrow $t: 1 \rightarrow \Omega$ in a topos: it represents the full subcategory spanned by the truth values, i.e. the subobjects of 1.

How do we externalize an internal full subcategory $\iota: \mathcal{E} \rightarrow B$ as a fibration over \mathcal{S} ? Look at the example of the subcategory \mathfrak{B} of \underline{Set} spanned by a family $\{X_b\}_{b \in B}$. If \mathfrak{B} is presented as $\iota: \mathcal{E} \rightarrow B$, the K -indexed families of objects from \mathfrak{B} can be obtained by pulling back ι along the functions $K \rightarrow B$. Together, the obtained arrows span a small full subcategory in the category \underline{Set}/K of K -indexed sets. In general, the externalisation of an internal full subcategory \mathfrak{B} given by $\iota: \mathcal{E} \rightarrow B$ in \mathcal{S} is the full subcategory \mathcal{J}/\mathcal{S} spanned in the arrow category \mathcal{S}/\mathcal{S} by the class \mathcal{J} of the arrows obtained by pulling back ι . This category is again fibred by the codomain functor and we write $\nabla \mathfrak{B}: \mathcal{J}/\mathcal{S} \rightarrow \mathcal{S}$. The elements of the class \mathcal{J} are called *extents*, and the arrow $\iota: \mathcal{E} \rightarrow B$ is *generic* for them. When \mathcal{S} is a topos and ι the truth arrow, \mathcal{J} is the class of monics. By pulling back the truth $t: 1 \rightarrow \Omega$ along a predicate $\varphi: K \rightarrow \Omega$, one indeed obtains the ordinary extent of φ .

Any family of arrows \mathcal{J} , closed under isomorphism, and stable under pullbacks can be seen as an *arrow fibration*, i.e. a subfibration $\mathcal{J}/\mathcal{S} \rightarrow \mathcal{S}$ of the basic fibration $\mathcal{S}/\mathcal{S} \rightarrow \mathcal{S}$. A *small full sub-category* of \mathcal{S} is thus an arrow fibration which has a generic arrow – and thus externalizes an internal subcategory.

The constructions obtained in the previous section for small categories go through for small full subcategories. If we replace \mathbb{B} by \mathfrak{B} , Theorem 2.3 goes through unchanged. Theorem 2.1 even takes a simpler form, due to the special role of the mixed exponents here. Namely, in an arrow fibration, the mixed exponent $[x, y] = x_* x^* y$, for $x, y \in \mathcal{J}/K$, yields both the exponent $[x, y]$ and the hom-set representant $\lfloor x, y \rfloor$. The proof of Theorem 2.3 – either part (A) or part (B) – can be reformulated using the mixed exponents everywhere; hence the following result.

3.2. Theorem. *Every small complete full subcategory is cocomplete.*

3.3. A small cocomplete category which is not complete. Unlike small categories, small full subcategories are not closed dualizing: the opposite of a subcategory is seldom a subcategory. When the base category \mathcal{S} is locally Cartesian closed, an internal full subcategory can be presented as an internal category – and then we can dualize it. The results of Section 2 then go through: the completeness and the cocompleteness are equivalent. However, when \mathcal{S} is *not* locally Cartesian closed – the equivalence is lost.

To get a counterexample, it suffices to spoil the Cartesian closed structure of the category \mathcal{S} of encoded sets. Restrict, for instance, the morphisms to those functions which are traced not by partial, but by primitive recursive functions. (Total would do as well.) Denote the resulting category by $\tilde{\mathcal{S}}$. It contains the internal subcategory of pers, described in 3.1, since its generic arrow $\iota: \mathcal{E} \rightarrow B$ is traced by the identity. The induced extents $\tilde{\mathcal{J}}$ are again just the indexed families of pers. Construction (39) of small coproducts, as well as the (omitted) construction of the finite fibrewise colimits, go through unchanged: the codes of functions play no role. So pers form a small cocomplete full subcategory \mathfrak{B} in $\tilde{\mathcal{S}}$.

On the other hand, if \mathfrak{B} were $\tilde{\mathcal{S}}$ -complete, it would have to be Cartesian closed: the mixed exponents $[\alpha, \beta]$ would be the exponents $[\alpha, \beta]$. Since 1 generates \mathfrak{B} , in order to satisfy $\mathfrak{B}(\alpha, \beta) \simeq \mathfrak{B}(1, [\alpha, \beta])$, these exponents should be in the form

$$|[\alpha, \beta]| = \mathfrak{B}(\alpha, \beta), \text{ with the encoding}$$

$$[\alpha, \beta](u) = \text{the codes tracing the function } u. \tag{43}$$

Now consider the set of natural numbers \mathbb{N} , with each number encoded by itself. This is a per. The exponent $[\mathbb{N}, \mathbb{N}]$ is

$$|[\mathbb{N}, \mathbb{N}]| = \text{primitive recursive functions;}$$

$$[\mathbb{N}, \mathbb{N}](u) = \text{the codes of the function } u. \tag{44}$$

To trace the evaluation $\varepsilon: [\mathbb{N}, \mathbb{N}] \times \mathbb{N} \rightarrow \mathbb{N}: \langle u, n \rangle \mapsto u(n)$, we need the universal primitive recursive code: a natural number e such that

$$e' \langle n, m \rangle = n'm. \tag{45}$$

But the diagonal argument shows that e' cannot be a total function – and certainly not primitive recursive. (If $k \in \mathbb{N}$ is the code for the function $\lambda x. e' \langle x, x \rangle + 1$, then $e' \langle k, k \rangle = k'k = e' \langle k, k \rangle + 1$.)

4. Around small full subcategories

4.1. Terminology. A small full subcategory \mathfrak{B} is said to be *comprehensive* if the corresponding class of extents \mathcal{J} contains all monics from the base category \mathcal{S} . If \mathfrak{B} is finitely complete and \mathcal{S} -small cocomplete, so that regular logic can be encoded in it, then “comprehensive” means that it must support the principle of function comprehension [35, 36].

We say that \mathfrak{B} is *exponentiable* if its object of objects B is: there is a functor $\wp: \mathcal{S}^{\text{op}} \rightarrow \mathcal{S}$, with a natural bijection $\mathcal{S}(I \times X, B) \simeq \mathcal{S}(I, \wp X)$. An exponentiable comprehensive category \mathfrak{B} is one of the main ingredients of the structure of *relative topos*, mentioned in the introduction. A finitely complete category \mathcal{S} is a topos if and only if it contains an exponentiable comprehensive small full subcategory, minimal in the sense that its extents are only monics and that it is skeletal, i.e. the classifiers are unique. Relative topos generally allows a larger class of extents and nonunique

classifiers. This relativization unhinges many fundamental topos-theoretical constructions. The point is now that they must be recovered in strongly constructive logic – perhaps with some more assumptions, and more work. Here is one of them: the promised generalization of Paré’s Theorem.

4.2. Theorem. *Let \mathcal{S} be finitely complete, \mathfrak{B} comprehensive and exponentiable. Then \mathcal{S} is finitely cocomplete if either*

(A) \mathfrak{B} or

(B) \mathcal{S}

is fibrationally complete.

Idea for the proof. When \mathfrak{B} is comprehensive, the exponentiation functor $\wp : \mathcal{S}^{\text{op}} \rightarrow \mathcal{S}$ is faithful [1, 5.1., Lemma 3]. On the other hand, \mathcal{S} must be well-powered, since the subobjects of K are weakly classified by the arrows $K \rightarrow B$. If \mathcal{S} is complete, Proposition 1.5(1) immediately yields the colimits. But in the most interesting examples – those derived from encoded sets and pers – \mathcal{S} is just finitely complete. However, it is fibrationally complete, as $\forall \mathcal{S}$. So the idea is to apply 1.5(1), but with the fibrational products instead of the ordinary ones. This will yield 4.2(B). And then it turns out that slight modifications make the fibrational products of \mathfrak{B} sufficient. Hence 4.2(A). This reduction is not contingent, but has to do with the relation of polymorphism and the dependent types, echoed in conditions (A) and (B) throughout the paper. Indeed, the conditions of 4.2 parallel those from 2.1, since the fibrational products and the fibrewise exponents are interdefinable in arrow fibrations [20, 2.5–6].

Proof of the Theorem. The fact that the functor \wp is faithful means that condition 1.5(a₁) is fulfilled. We want to reformulate in fibrational terms the proof of (a₁) \Leftrightarrow (b₁), for the special case of diagram (3) with a finite category \mathbb{C} . Just as in Section 1, sending a diagram $D \in \mathcal{S}^{\mathbb{C}}$ around (3) yields a cocone $\omega = \omega_D : D \rightarrow \Delta L$. Every other cocone $d : D \rightarrow \Delta A$ factorizes uniquely through a subobject S of the target $L = L_D = \wp(\varprojlim \wp D)$ of ω . The crucial part of (15) is now as follows:

$$\begin{array}{ccc}
 D & \xrightarrow{\omega} & \Delta L = \Delta \wp \varprojlim \wp D \\
 \downarrow d & \searrow h & \downarrow \Delta s \\
 \Delta S & \xrightarrow{\Delta s} & \Delta L \\
 \downarrow \Delta d' & \lrcorner & \downarrow \\
 \Delta A & \xrightarrow{\Delta \eta} & \Delta \wp \wp A
 \end{array}
 \tag{46}$$

A coinitial set of upper bounds for D is contained among the subobjects of L , and hence in the set Σ of the extents with the codomain L . This is a solution set for D . It is internally represented by $\wp L$: each extent over L corresponds to some arrow $1 \rightarrow \wp L$. The product of a family indexed by Σ in formula (10) can now be replaced by the fibrational product of an internal family indexed by $\wp L$. Beginning with the cocone $\omega: D \rightarrow \Delta L$, we construct a weakly universal cocone $\delta: D \rightarrow WD$

$$\delta = \prod_{s \in \mathcal{S}/L} [\lim_{\leftarrow} [\omega, s], s], \tag{47}$$

where s runs over the extents with the given codomain. The construction of δ is thus performed in two steps: first send D around (3) to get $L := \wp(\lim_{\leftarrow} \wp D)$ and ω ; and then send ω around another adjoint square, similar to (9). This time, case (B) is simpler.

(B) When \mathcal{S} is locally Cartesian closed, we get δ using the following square:

$$\begin{array}{ccc}
 (\mathcal{S}/\wp L \times L)^{op} & \xrightarrow{\Delta^{op}} & ((\mathcal{S}/\wp L \times L)^{op})^{c^{op}} \\
 \uparrow \scriptstyle Q_! & \swarrow \scriptstyle \tau & \downarrow \scriptstyle \lim^{op} \\
 & & ((\mathcal{S}/\wp L \times L)^{op})^c \\
 \uparrow \scriptstyle Q_! & & \uparrow \scriptstyle Q_! \circ (-) \\
 \mathcal{S}/L & \xrightarrow{\Delta} & (\mathcal{S}/L)^c
 \end{array}
 \tag{48}$$

where

$$Qa := \pi_* [a, \exists],$$

$$Q_!b := [\pi^*b, \exists].$$

The arrow $\pi: \wp L \times L \rightarrow L$ is the projection, and $\exists = \exists_L \in \mathcal{S}/\wp L \times L$ is the generic extent. As the readers familiar with topos theory will know, \exists is obtained by pulling back the generic arrow $\iota: \mathcal{E} \rightarrow B$ along the evaluation $e: \wp L \times L \rightarrow B$. Going around (48) yields the functor

$$V: (\mathcal{S}/L)^c \rightarrow \mathcal{S}/L: c \mapsto Q(\lim_{\leftarrow} Q_!c). \tag{49}$$

Every $c \in (\mathcal{S}/L)^c$ – which is a cocone $c: C \rightarrow \Delta L$ from some diagram $C: \mathbb{C} \rightarrow \mathcal{S}$ – induces a cocone $\delta_c \in (\mathcal{S}/L)^c(c, \Delta Vc)$, such that $c = \Delta Vc \circ \delta_c$. Namely, δ_c is the unit of the adjunction $\lim_{\leftarrow} Q_! \circ (-) \dashv \Delta Q$. By Lemma 4.3 (with $\mathcal{X} = \mathcal{S}$), this decomposition of c is weakly initial for all decompositions in the form $c = \Delta s \circ h$, where s is an extent, and h an arbitrary cocone. (Vc itself may not be an extent.) Now we can conclude that δ_ω is weakly initial among the cocones from D . First of all, (46) shows that every cocone $d: D \rightarrow \Delta A$ factorizes as $d = \Delta d' \circ h$ through a cocone $h: D \rightarrow \Delta S$ such that $\omega = \Delta s \circ h$, for some extent (even monic) $s: S \rightarrow \Delta L$. Since $\omega = \Delta V\omega \circ \delta_\omega$ is initial for

such factorisations, there must be an arrow h' , such that $h = \Delta h' \circ \delta_\omega$. Hence, arbitrary cocone $d: D \rightarrow \Delta A$ decomposes as $d = \Delta d' \circ \Delta h' \circ \delta_\omega$. In other words, $\delta = \delta_\omega$ is a weak colimit.

The strong colimit can be obtained just as in 1.2. Of course, the mixed exponent, occurring there, must be internalized, but that is routine.

(A) Now we shall assume that only the fibrational products of the extents exist and restrict the functors Q and $Q_!$ (48) from \mathcal{S}/L to \mathcal{J}/L . Hence the functors R and $R_!$ (50). To extend them to all of \mathcal{S}/L , again, we use Theorem 3.2, i.e. the fact that \mathcal{L} is cocomplete, since it is complete by assumption. The cocompleteness implies that every fibre \mathcal{J}/K is a reflective subcategory of \mathcal{S}/K . Hence the functors U and $U_!$.

$$\begin{array}{ccc}
 (\mathcal{S}/\wp L \times L)^{op} & \xrightarrow{\Delta^{op}} & ((\mathcal{S}/\wp L \times L)^{c^{op}})^{op} \\
 \uparrow & \Uparrow & \uparrow \\
 & \lim^{op} & \\
 & \cong & \\
 & ((\mathcal{S}/\wp L \times L)^{op})^c & \\
 \begin{array}{c} \uparrow \\ R, \dashv \\ \downarrow R \\ \mathcal{J}/L \end{array} & & \begin{array}{c} \uparrow \\ R_! \circ (-) \dashv \\ \downarrow R \circ (-) \\ (\mathcal{J}/L)^c \end{array} \\
 \begin{array}{c} \uparrow \\ U_! \dashv \\ \downarrow U \\ \mathcal{S}/L \end{array} & & \begin{array}{c} \uparrow \\ U_! \circ (-) \dashv \\ \downarrow U \circ (-) \\ (\mathcal{S}/L)^c \end{array} \\
 & \xrightarrow{\Delta} & (\mathcal{S}/L)^c
 \end{array} \tag{50}$$

The functors are

$$\begin{aligned}
 Ra &:= \pi_* \lceil a, \exists \rceil, \\
 R_!(\beta) &:= \lceil \pi^* \beta, \exists \rceil, \\
 U(\beta) &:= \beta, \\
 U_! b &:= b_!(id).
 \end{aligned}$$

Going around this diagram, we define the functor

$$W: (\mathcal{S}/L)^c \rightarrow \mathcal{S}/L: c \mapsto UR(\lim_{\leftarrow} R_! U_! c) \tag{51}$$

Just as under (B), every cocone $c: C \rightarrow \Delta L$ decomposes into $c = \Delta Wc \circ \rho_c$, where ρ_c is the unit of the adjunction $\lim_{\leftarrow} Q_! \circ (-) \dashv \Delta Q$. By Lemma 4.4 (with $\mathcal{X} = \mathcal{J}$), this decomposition is weakly initial among the factorisations $c = \Delta s \circ h$, where s is an extent, and h an arbitrary cocone. (Unlike Vc , the arrow Wc is always an extent.) Exactly as before, $\delta = \rho_\omega$ yields a colimit of D . \square

Let \mathcal{S} be a finitely complete category with exponentiable small subcategory \mathfrak{B} . Let \mathcal{X} be a family of arrows in \mathcal{S} , such that $\nabla \mathcal{X} : \mathcal{X}/\mathcal{S} \rightarrow \mathcal{S}$ is a complete fibration and $\mathcal{J} \subseteq \mathcal{X}$. (In fact, only the cases $\mathcal{X} = \mathcal{J}$ and $\mathcal{X} = \mathcal{S}$ will matter.) Consider the scheme

$$\begin{array}{ccc}
 (\mathcal{S}/\wp L \times L)^{\text{op}} & \xrightarrow{\Delta^{\text{op}}} & ((\mathcal{S}/\wp L \times L)^{\mathbb{C}^{\text{op op}}}) \\
 & \xleftarrow{\Gamma} & \\
 & \xleftarrow{\lim^{\text{op}}} & \cong \\
 & & ((\mathcal{S}/\wp L \times L)^{\text{op}})^{\mathbb{C}} \\
 \uparrow Q, \downarrow Q & & \uparrow Q, \circ(-), \downarrow Q, \circ(-) \\
 \mathcal{X}/L & \xrightarrow{\Delta} & (\mathcal{X}/L)^{\mathbb{C}}
 \end{array} \tag{52}$$

where \mathbb{C} is a finite category, and

$$Qa := \pi_* \lceil a, \ni \rceil,$$

$$Qb := \lceil \pi^* b, \ni \rceil.$$

In the familiar way, going around (52) yields a functor

$$V : (\mathcal{X}/L)^{\mathbb{C}} \rightarrow \mathcal{X}/L : c \mapsto Q(\lim_{\leftarrow} Qc) = \pi_* \lceil \lim_{\leftarrow} \lceil \pi^* c, \ni \rceil, \ni \rceil \tag{53}$$

and every \mathcal{X} -cocone $c \in (\mathcal{X}/L)^{\mathbb{C}}$ induces an \mathcal{X} -cocone $\delta_c \in (\mathcal{X}/L)^{\mathbb{C}}(c, \Delta Vc)$ as the unit of the adjunction $\lim_{\leftarrow} Q, \circ(-) \dashv \Delta Q$. We claim that δ_c is also the unit of some kind of a partial weak adjunction of V and Δ .

4.3. Lemma. *Let $c : C \rightarrow \Delta L$ be an \mathcal{X} -cocone and $s : S \rightarrow L$ an extent. For every \mathcal{X} -cocone $h \in (\mathcal{X}/L)^{\mathbb{C}}(c, \Delta s)$, there is an arrow $h' \in \mathcal{X}/L(Vc, s)$ such that $h = \Delta h' \circ \delta_c$.*

$$\begin{array}{ccc}
 & & h \\
 & \curvearrowright & \\
 C & \xrightarrow{\delta_c} & \Delta S \\
 & \dashrightarrow \Delta h' & \\
 & & \\
 c & \searrow & \Delta s \\
 & \Delta Vc & \\
 & \searrow & \\
 & \Delta L &
 \end{array} \tag{54}$$

Proof. We shall construct

(i) an arrow $p : Vc \rightarrow \lceil \lim_{\leftarrow} \lceil c, s \rceil, s \rceil$ in \mathcal{X}/L for any given extent s and \mathcal{X} -cocone c ;

(ii) an arrow $q : \lceil \lim_{\leftarrow} \lceil c, s \rceil, s \rceil \rightarrow s$ in \mathcal{X}/L for any extent s and \mathcal{X} -cocones c and h .

Finally, we shall show that these data yield the required decomposition in the form

(iii) $h' := q \circ p$.

(i) Just as in a topos, the generic extnt $\exists \in \mathcal{S}/\wp L \times L$, classified by the evaluation $e: \wp L \times L \rightarrow B$, classifies the extents over L . Namely, if the extnt $s: S \rightarrow L$ is $s = \zeta^*i$, then $s = \langle s'', id \rangle^* \exists$ holds too, for $s'' = s' \circ \phi$, where $s': 1 \rightarrow \wp L$ is the transpose of $\zeta: L \rightarrow B$, while $\phi: L \rightarrow 1$ is the terminal arrow. Using s'' , we derive p

$$p := \frac{\varepsilon: \pi^* \pi_* \lceil \lim[\pi^*c, \exists], \exists \rceil \rightarrow \lceil \lim[\pi^*c, \exists], \exists \rceil}{\langle s'', id \rangle^* \varepsilon: \langle s'', id \rangle^* \pi^* \pi_* \lceil \lim[\pi^*c, \exists], \exists \rceil \rightarrow \langle s'', id \rangle^* \lceil \lim[\pi^*c, \exists], \exists \rceil}$$

Since the exponents and the limits are stable, $\langle s'', id \rangle^* \exists = s$ implies

$$\langle s'', id \rangle^* \lceil \lim[\pi^*c, \exists], \exists \rceil = \lceil \lim[c, s], s \rceil.$$

On the other side, for suitable inverse images (or up to iso), the equation $\pi \circ \langle s'', id \rangle = id$ implies

$$\langle s'', id \rangle^* \pi^* \pi_* \lceil \lim[\pi^*c, \exists], \exists \rceil = \pi_* \lceil \lim[\pi^*c, \exists], \exists \rceil = Vc.$$

(ii) By the componentwise transposition, a cocone $h: c \rightarrow s$ in \mathcal{X}/L induces a cone $h': id \rightarrow \lceil c, s \rceil$. This cone factors through the limit of $\lceil c, s \rceil: \mathbb{C}^{op} \rightarrow \mathcal{X}/L$ by $\lceil h \rceil: id \rightarrow \lim[\lceil c, s \rceil]$. The arrow $q: \lceil \lim[\lceil c, s \rceil], s \rceil \rightarrow s$ is then defined by evaluating $\lceil \lim[\lceil c, s \rceil], s \rceil$ at $\lceil h \rceil$.

$$q := \frac{\frac{\frac{h: c \rightarrow s}{h': id \rightarrow \lceil c, s \rceil}}{\lceil h \rceil: id \rightarrow \lim[\lceil c, s \rceil]}}{h_{\lceil c, s \rceil}: \lceil \lim[\lceil c, s \rceil], s \rceil \rightarrow id \rightarrow \lim[\lceil c, s \rceil]}}$$

$$q := e_{\lceil c, s \rceil} \circ \langle id, h_{\lceil c, s \rceil} \rangle: \lceil \lim[\lceil c, s \rceil], s \rceil \rightarrow \lceil \lim[\lceil c, s \rceil], s \rceil \times \lim[\lceil c, s \rceil] \rightarrow s$$

(In subscripts, ℓ abbreviates $\lim[\lceil c, s \rceil]$.)

(iii) The following calculation shows that the arrow $h' = q \circ p$ satisfies the equation $\Delta h' \circ \delta_c = h$.

$$\Delta h' \circ \delta_c = \Delta q \circ \Delta p \circ \delta_c \stackrel{(a)}{=} \Delta q \circ \gamma_s = \Delta e_{\ell} \circ \Delta \langle id, h_{\lceil c, s \rceil} \rangle \circ \gamma_s$$

$$\stackrel{(b)}{=} e_c \circ (g_s \times id) \circ \langle h_c, id \rangle \stackrel{(c)}{=} h. \tag{55}$$

To justify step (a), notice that for any diagram $c: \mathbb{C} \rightarrow \mathcal{X}/L$ and any extnt s over L , the limit cone $g_s: \lim[\lceil c, s \rceil] \rightarrow \lceil c, s \rceil$ induces a cocone $\gamma_s: c \rightarrow \lceil \lim[\lceil c, s \rceil], s \rceil$ by two componentwise transpositions. If s is \exists and if c is π^*c , a further transposition of $\gamma_{\exists}: \pi^*c \rightarrow \lceil \lim[\pi^*c, \exists], \exists \rceil$ yields δ_c .

$$\frac{g_{\exists}: \lim[\pi^*c, \exists] \rightarrow \lceil \pi^*c, \exists \rceil,}{g'_{\exists}: \lim[\pi^*c, \exists] \times \pi^*c \rightarrow \exists,}$$

$$\frac{\gamma_{\exists}: \pi^*c \rightarrow \lceil \lim[\pi^*c, \exists], \exists \rceil,}{\delta_c: c \rightarrow \pi_* \lceil \lim[\pi^*c, \exists], \exists \rceil.}$$

This derivation explains the commutativity of the triangle in the following diagram of cocones. By chasing it, we get $\Delta p \circ \delta_c = \gamma_s$.

$$\begin{array}{ccc}
 c & \xrightarrow{\delta_c} & \pi_* \lceil \lim \lceil \pi^* c, \exists \rceil, \exists \rceil \\
 \downarrow = & & \downarrow = \\
 \langle s'', id \rangle * \pi^* c & \xrightarrow{\langle s'', id \rangle * \pi^* \delta_c} & \langle s'', id \rangle * \pi^* \pi_* \lceil \lim \lceil \pi^* c, \exists \rceil, \exists \rceil \\
 \downarrow = & \searrow \langle s'', id \rangle * \gamma_s & \downarrow p = \langle s'', id \rangle * \epsilon \\
 c & & \langle s'', id \rangle * \lceil \lim \lceil \pi^* c, s \rceil, \exists \rceil \\
 & \searrow \gamma_s & \downarrow = \\
 & & \lceil \lim \lceil c, s \rceil, s \rceil
 \end{array} \tag{56}$$

Step (b) of (55) is based on yet another diagram of cocones. Similarly as in (ii), the cocone h_c is $\lceil h^\top \circ \phi_c : c \rightarrow id \rightarrow \lim \lceil c, s \rceil$. (This time, the terminal arrow $\phi_c : c \rightarrow id$ is the cocone c itself.)

$$\begin{array}{ccccc}
 c & \xrightarrow{\langle h_c, id \rangle} & & \lim \lceil c, s \rceil \times c & \\
 \downarrow \gamma_s & \searrow \langle id, h_c \rangle & & \downarrow g_s \times id & \\
 \lceil \lim \lceil c, s \rceil, s \rceil & & c \times \lim & \lceil c, s \rceil \times c & \\
 \downarrow \langle id, h_{\lceil c, s \rceil} \rangle & \swarrow \gamma_s \times id & & \downarrow e_c & \\
 \lceil \lim \lceil c, s \rceil, s \rceil \times \lim \lceil c, s \rceil & \xrightarrow{e_\lceil} & & s &
 \end{array} \tag{57}$$

Finally, to justify (c), recall that $\lceil h^\top : id \rightarrow \lim \lceil c, s \rceil$ has been defined as the unique arrow satisfying $g_s \circ \lceil h^\top = h'$, where $g_s : \lim \lceil c, s \rceil \rightarrow \lceil c, s \rceil$ is the limit cone and $h' : id \rightarrow \lceil c, s \rceil$ is the componentwise transpose of the cocone $h : c \rightarrow s$. The equation $e_c \circ \langle h' \circ \phi_c, id \rangle = h$ thus implies $e_c \circ \langle g_s \circ h_c, id \rangle = h$. \square

Now suppose that the complete fibration $\nabla \mathcal{X}$, considered in 4.3, is cocomplete too, so that the inclusion $U : \mathcal{X}/L \hookrightarrow \mathcal{S}/L$ has a left adjoint

$$U_! : \mathcal{S}/L \rightarrow \mathcal{X}/L : b \mapsto b_!(id).$$

Hence the functor

$$W = VU_! : (\mathcal{S}/L)^C \rightarrow \mathcal{X}/L : c \mapsto \pi_* \lceil \lim \lceil \pi^* c_!(id), \exists \rceil, \exists \rceil. \tag{58}$$

Each cocone $c \in (\mathcal{S}/L)^c$ induces a cocone

$$\rho_c = \delta_c \circ \sigma_c \in (\mathcal{S}/L)^c(c, \Delta U W c), \tag{59}$$

where $\sigma_c \in (\mathcal{S}/L)^c(c, U U_1 c)$ is the natural transformation obtained by restricting the unit of $U_1 \dashv U$ to the vertices of c , while δ_c is constructed as in 4.3.

4.4. Lemma. *Let $c : C \rightarrow \Delta L$ be any cocone and $s : S \rightarrow L$ an extant. For every cocone $h \in (\mathcal{S}/L)^c(c, \Delta s)$, there is an arrow $h' \in \mathcal{X}/L(Wc, s)$ such that $h = \Delta h' \circ \rho_c$.*

Proof. Every component $hi \in \mathcal{S}/L(ci, s)$ of h corresponds to a unique arrow $\tilde{h}i \in \mathcal{X}/Ci(id, ci^*s)$, and further, by the adjunction $c_1 \dashv c^*$, to an arrow $\tilde{h}i \in \mathcal{X}/L(ci, (id), s)$, so that $hi = \tilde{h}i \circ \sigma_{ci}$. In this way, the cocone $h \in (\mathcal{S}/L)^c(c, \Delta s)$ induces a cocone $\tilde{h} \in (\mathcal{X}/L)^c(U_1c, \Delta s)$, with $h = \tilde{h} \circ \sigma_c$. Lemma 4.3 now yields $h' \in \mathcal{X}/L(Wc, s)$, with $\tilde{h} = \Delta h' \circ \delta_c$. Hence, as required

$$h = \tilde{h} \circ \sigma_c = \Delta h' \circ \delta_c \circ \sigma_c = \Delta h' \circ \rho_c. \quad \square$$

5. Around small categories

5.1. If \mathcal{S} is locally Cartesian closed, every internal full subcategory \mathfrak{B} can be described as an internal category \mathbb{B} : the pair $\langle \partial_0, \partial_1 \rangle : B_1 \rightarrow B \times B$ is the exponent $[\iota_0, \iota_1]$, where ι_0 and ι_1 are obtained by pulling back ι along the projections π_0 and $\pi_1 : B \times B$. This is not hard to understand when \mathcal{S} is Set.

The other way around, each “object” $x : 1 \rightarrow B$ of an internal category \mathbb{B} in \mathcal{S} determines an internal full subcategory \mathfrak{B} : the generic arrow $\iota : \mathfrak{E} \rightarrow B$ is obtained by pulling back $\partial_0 : B_1 \rightarrow B$ along $x : 1 \rightarrow B$, and then postcomposing the obtained inclusion $\mathfrak{E} \rightarrow B_1$ with ∂_1 . The idea is that each object b of \mathbb{B} should be represented by the set $\mathbb{B}(x, b)$, displayed in the fibre of ι over b . When x generates \mathbb{B} , this representation is faithful. We say that x *fully generates* \mathbb{B} if this representation is full and faithful. A small category \mathbb{B} with a terminal object \top is equivalent to a small full subcategory \mathcal{L} with a terminal object if and only if \top fully generates \mathbb{B} . This is true for fibrations in general [30, III.4.3].

In this section, we want to extend Theorem 4.2 to a setting with \mathbb{B} instead of \mathfrak{B} . The difference is logically significant. If \mathcal{S} is thought of as the category of “sets” and if \mathbb{B} is the category of “propositions”, then saying that \mathbb{B} is (equivalent to) a full subcategory of \mathcal{S} means that propositions are just special sets and that logic is *extensional*. In general, this is not the case. For instance, an internal functor category \mathbb{B}^c is never generated by its terminal object, unless \mathbb{C} is trivial. On the other hand, it is complete as soon as \mathbb{B} is. Nevertheless, every small category \mathbb{B} induces a small full subcategory: as outlined above, each proposition $b \in \mathbb{B}$ can be represented by the set of its proofs $\mathbb{B}(\top, b)$, even if not faithfully. The small full subcategory derived from \mathbb{B} in this way is its *extensional collapse*. We shall denote it $\iota\mathbb{B}$, as in [30]. It can be a very bad approximation of \mathbb{B} , but it does inherit some logical structure.

5.2. Proposition. [30, III.4.1] *If a small category \mathbb{B} is complete, its extensional collapse is complete and cartesian closed.*

Proof. As explained above, $\imath\mathbb{B}$ is represented by the arrow

$$\imath := \partial_1 \circ \partial_0^* \top : \Xi \rightarrow B_1 \rightarrow B. \tag{60}$$

and externalized as the fibration $\mathcal{J}/\mathcal{S} \rightarrow \mathcal{S}$. A choice of an extent $\imath\Psi = \Psi^* \imath \in \mathcal{J}/K$ for each object $\Psi : K \rightarrow B$ of \mathcal{S}/\mathbb{B} determines the *comprehension functor* $\imath : \mathcal{S}/\mathbb{B} \rightarrow \mathcal{J}/\mathcal{S}$, clearly Cartesian. If Ψ is thought of as a predicate over K , its extent $\imath\Psi$ is $\{x \in K \mid \Psi(x)\} \rightarrow K$. The fibre of $\imath\Psi$ over $a \in K$ is the set of the proofs of $\Psi(a)$. (If the proofs are unique, the extent is monic.) This idea is formally expressed by the natural correspondence

$$\mathcal{S}/K(a, \imath\Psi) \simeq \mathbb{B}_A(\top A, a^* \Psi), \tag{61}$$

which reader may wish to check. The object A is the domain of a , $\top A$ is the terminal object in the fibre \mathbb{B}_A .

Using (61), we prove that the comprehension functor induces the small products in $\imath\mathbb{B}$:

$$f_*(\imath\Psi) := \imath(f_* \Psi). \tag{62}$$

The Beck–Chevalley condition (B) and the stability of the terminal objects (S) are also used.

$$\begin{aligned} \mathcal{J}/L(\imath\varphi, \imath(f_* \Psi)) &\stackrel{(61)}{\simeq} \mathbb{B}_{D\varphi}(\top D\varphi, \imath\varphi^* f_* \Psi) \stackrel{(B)}{\simeq} \mathbb{B}_{D\varphi}(\top D\varphi, \imath_*(f^* \imath\varphi)^* \Psi) \\ &\simeq \mathbb{B}_{Df^*\varphi}(\imath^* \top D\varphi, (f^* \imath\varphi)^* \Psi) \stackrel{(S)}{\simeq} \mathbb{B}_{Df^*\varphi}(\top Df^*\varphi, (f^* \imath\varphi)^* \Psi) \\ &\stackrel{(61)}{\simeq} \mathcal{J}/K(f^* \imath\varphi, \imath\Psi). \end{aligned}$$

The domain of $\imath\varphi$ is here written $D\varphi$, rather than $\{x \mid \varphi(x)\}$. To show that the functor \imath also induces the fibrewise limits in $\imath\mathbb{B}$, observe that the functors $D = \text{Dom} \circ \imath$, $\nabla\mathbb{B} = \text{Cod} \circ \imath$ and \top (the fibrewise terminal objects) are connected by the adjunctions $\nabla\mathbb{B} \dashv \top \dashv D$, with $D \circ \top = \text{Id} = \nabla\mathbb{B} \circ \top$. Hence

$$\imath\varphi \simeq D\eta_\varphi, \tag{63}$$

where $\eta_\varphi : \varphi \rightarrow \top \nabla\mathbb{B}(\varphi)$ is the unit of the latter adjunction. It follows that

$$\imath(\lim_{\leftarrow} \Gamma) \simeq D(\lim_{\leftarrow} \Gamma_\eta) \simeq \lim_{\leftarrow} D\Gamma_\eta \simeq \lim_{\leftarrow} \imath\Gamma, \tag{64}$$

where Γ_η is the diagram of terminal arrows induced by the diagram Γ .

Together, (62) and (64) say that $\imath\mathbb{B}$ is complete, and that the comprehension functor preserves limits. As any full subcategory, $\imath\mathbb{B}$ must be Cartesian closed whenever it is complete: the exponents are $[x, y] = x_* x^* y$. The exponents in \mathbb{B} may not exist; and if they do, they will be preserved under the comprehension only if it is an equivalence [30, III.4.3]. \square

Remark. The comprehension functors were introduced in Lawvere’s seminal paper [24], for *elementary doctrines*. In a more general setting, they have been studied in

[30], together with the notion of extensional collapse. Only for the sake of simplicity, these notions have been restricted to small fibrations here.

By definition, a small category is comprehensive and exponentiable if its extensional collapse is. As an immediate consequence of Theorem 4.2 and Proposition 5.2, we get the following theorem.

5.3. Theorem. *Let \mathcal{S} be finitely complete category and \mathbb{B} comprehensive and exponentiable. Then \mathcal{S} is finitely cocomplete if either*

(A) \mathbb{B} or

(B) \mathcal{S}

is fibrationally complete.

6. Conclusions

Type-theoretical constructions that can be derived from the results of this paper should be formulated in two-sorted type systems: one sort for “sets” \mathcal{S} , the other for “propositions” \mathbb{B} (or \mathfrak{B}). A typical example is the Theory of Constructions [7, 20]. The distinction between the extensional and the nonextensional case (\mathfrak{B} vs. \mathbb{B}) has been studied in [31]. The categorical semantics has been spelled out in [43, 30, ch. IV]. In this setting, Sections 2 and 3 can be understood as results about the interdefinability of the polymorphic sums and products in the presence of equality types. Sections 4 and 5, on the other hand, offer an effective treatment of some fundamental concepts which have, so far, not been represented in the bestiary of type theory. Paré’s Theorem allows, for instance, an implementation of the quotients in terms of the exponents and the equalizers, avoiding the ineffectiveness of the transitive closure. We believe that basic set-theoretical notions, including those uncovered in topos theory, should be available in computation, and accounted for in any foundational structure.

Last but not the least, this paper demonstrates how familiar logical ideas grow into complicated formulas, when enriched with proofs-as-constructions. This fact is well-known from type theory; we just slightly extended it by the use of category theory. Irony aside, a royal way to strongly constructive logic probably does not exist. In principle, the categorical approach should *simplify* the picture (although, of course, nothing can stop an author – including the present one – from producing unnecessarily complicated arguments). By describing the *logical-operations-as-universal-properties* (i.e., adjunctions [24]) category theory drains the flood of structure and allows the underlying fluvial systems to reappear.² In this way, it might provide some

² Although it is an offspring of intuitionism, strongly constructive logic often lacks intuitive leads – and category theory supplies some. For instance, there was no logical experience to suggest the correct reduction rules for the proofs of the existentially quantified formulas; one of them was discovered only by following an adjunction [19, Remark 1.9(i)]. A similar story can be told about the function comprehension [35–36].

“conceptual mathematics”, which the ongoing syntactical studies of foundations [17] seem to be calling for.

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