Calculemus Igitur

Lambert Meertens
Kestrel Institute & Utrecht University

$\mathrm{I} \heartsuit \oplus / \cdot f *$

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1986: Theory of Lists

$$f* \cdot \mathbin{+\!\!+}/ \quad = \quad \mathbin{+\!\!+}/ \cdot f**$$

$$\oplus/ \cdot \mathbin{+\!\!+}/ \quad = \quad \oplus/ \cdot \oplus/*$$

1997: Algebra of Programming

$$\Lambda R \;=\; (\in \backslash R) \cap (R \backslash \in)^\circ$$

$$(\![R]\!) \cdot (\![S^\circ]\!)^\circ \cdot (\![S^\circ]\!) \cdot (\![R]\!)^\circ \subseteq id$$

Binary Structures over A

Two formative operations:

$$\begin{aligned} Tip &:: \quad A \quad \rightarrow S_A \\ Fork &:: S_A \times S_A \rightarrow S_A \end{aligned}$$

possibly with algebraic laws

The Boom Hierarchy

Laws for *Fork*	Inhabitants of S_A
(none)	Tiptrees
Assoc	Lists
Assoc+Comm	Bags
Assoc+Comm+Idemp	Sets

But what about . . .

Laws for *Fork*

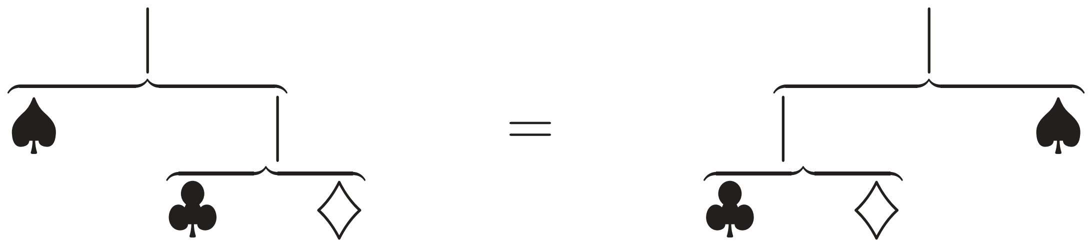
Comm

Inhabitants of S_A

Mobiles

?

For example,



Notation for Mobiles

$$\mathit{Tip}\ a \longrightarrow [a]$$

$$\mathit{Fork}(s, t) \longrightarrow s \frown t$$

The usual homomorphisms

$$
\begin{aligned}
f * [a] &\quad=\quad [f\ a] \\
f * (s \frown t) &\quad=\quad (f * s) \frown (f * t)
\end{aligned}
$$

For *symmetric* operator $\oplus$:

$$
\begin{aligned}
\oplus / [a] &\quad=\quad a \\
\oplus / (s \frown t) &\quad=\quad (\oplus / s) \oplus (\oplus / t)
\end{aligned}
$$

Examples

$shape \hat{=} \; !*$, where $! :: A \to 1$

$sum \hat{=} \; +/$

(for a mobile with numeric tips)

Catamorphisms

The general catamorphism on mobiles:

$$\oplus / \cdot f*$$

For example, if function $tipweight :: A \rightarrow \mathbb{R}_+$ gives the weights of the tree tips, function

$$treeweight \triangleq +/ \cdot tipweight*$$

returns the total weight of a mobile

Weight-balanced mobiles

A mobile is called (*weight-*)balanced when in each sibling pair of subtrees the siblings have the same treeweight

(Note. No relationship to *depth-balanced* unless all tips have the same weight)

Weight-balanced mobiles (2)

$$
\begin{array}{lcl}
\mathit{balanced}\,[a] & = & \mathrm{True} \\
\mathit{balanced}\,(s \hat{}\, t) & = & \\
\quad\quad \mathit{balanced}\,s & & \\
\quad \wedge\ \mathit{balanced}\,t & & \\
\quad \wedge\ \mathit{treeweight}\,s = \mathit{treeweight}\,t & &
\end{array}
$$

Zygomorphism

(Malcolm, 1990) Yet another origami pattern having both paramorphisms and "banana split" as special instances

$$(\mathit{balanced}, \mathit{treeweight}) \;=\; \oplus/\cdot f*$$

for some $\oplus$ and f

How to find $\oplus$ and f? Calculate!

Finding f

Since $(\oplus/ \cdot f*)\,[a] \;=\; f\ a$,

$$
\begin{aligned}
& f\ a \\
= & \quad \{\text{above, zygo}\} \\
& (\mathit{balanced}, \mathit{treeweight})\,[a] \\
= & \quad \{\text{commatics}\} \\
& (\mathit{balanced}\,[a],\ \mathit{treeweight}\,[a]) \\
= & \quad \{\text{definitions}\} \\
& (\text{True},\ \mathit{tipweight}\ a)
\end{aligned}
$$

Finding $\oplus$

Given

$$\begin{aligned} balanced\ s &= p & treeweight\ s &= u \\ balanced\ t &= q & treeweight\ t &= v \\ balanced(s \frown t) &= r & treeweight(s \frown t) &= w \end{aligned}$$

solve for $\oplus$

$$(p,u) \oplus (q,v) \ = \ (r,w)$$

Finding $\oplus$ (2)

$$
\begin{array}{ll}
 & r \\
= & \quad \{\text{given}\} \\
 & balanced(s \frown t) \\
= & \quad \{\text{definition}\} \\
 & balanced\ s \ \wedge \ balanced\ t \\
 & \quad \wedge \ treeweight\ s = treeweight\ t \\
= & \quad \{\text{given}\} \\
 & p \ \wedge \ q \ \wedge \ u = v
\end{array}
$$

Finding $\oplus$ (3)

Similarly, we find $w = u + v$, resulting in the definition

$$(p, u) \oplus (q, v) \;=\; (p \wedge q \wedge u = v,\; u + v)$$

Sanity check: $\oplus$ is indeed symmetric

Balancing mobiles

Consider mobiles over $\mathbb{R}_+$, where the tip values are the tip weights (i.e., *tipweight* is the identity function)

Given a weight (value) and a mobile, can we construct another mobile of the same shape that is balanced and has the given weight?

Balancing mobiles (2)

Abbreviate $(bal, tw) \hat{=} (balanced, treeweight)$

Find function *mkbal* satisfying

$$shape\ \ (w\ \text{`}mkbal\text{`}\ \ t) \ = \ shape\ t$$
$$(bal, tw)\,(w\ \text{`}mkbal\text{`}\ \ t) \ = \ (\mathrm{True},\ w)$$

Finding *mkbal*

From the preservation of *shape* we have

$$w \ \text{`}mkbal\text{`} \ [a] \quad = \ [b]$$
$$w \ \text{`}mkbal\text{`} \ (s\text{^}t) \ = \ x\text{^}y$$

for some b, x and y such that
$shape\,x = shape\,s$ and $shape\,y = shape\,t$

For which values? Calculemus!

Finding b

$$
\begin{array}{ll}
 & (bal, tw)[b] \;=\; (\text{True}, w) \\
\equiv & \qquad \{\text{definition}\} \\
 & (\text{True}, b) \;=\; (\text{True}, w) \\
\equiv & \qquad \{\text{commatics}\} \\
 & b \;=\; w
\end{array}
$$

Finding x and y

$$
\begin{aligned}
& (\mathit{bal}, \mathit{tw})\,(x \frown y) \;=\; (\mathrm{True}, w) \\
\equiv \quad & \{\text{definition}\} \\
& (\mathit{bal}\,x \wedge \mathit{bal}\,y \wedge \mathit{tw}\,x = \mathit{tw}\,y,\ \mathit{tw}\,x + \mathit{tw}\,y) \;= \\
& \qquad (\mathrm{True}, w) \\
\equiv \quad & \{\text{commatics, logic}\} \\
& \mathit{bal}\,x \wedge \mathit{bal}\,y \wedge \mathit{tw}\,x = \mathit{tw}\,y \wedge \\
& \qquad \mathit{tw}\,x + \mathit{tw}\,y = w
\end{aligned}
$$

Finding x and y (continued)

$$
\begin{array}{ll}
 & bal\,x \wedge bal\,y \wedge tw\,x = tw\,y \wedge \\
 & \qquad tw\,x + tw\,y = w \\
\equiv & \qquad \{\text{algebra}\} \\
 & bal\,x \wedge bal\,y \wedge tw\,x = tw\,y = w/2 \\
\equiv & \qquad \{\text{commatics}\} \\
 & (bal, tw)\,x = (\mathrm{True}, w/2) \wedge \\
 & (bal, tw)\,y = (\mathrm{True}, w/2) \\
\Leftarrow & \qquad \{\text{constructive hypothesis}\} \\
 & x = (w/2)\,\grave{}mkbal\grave{}\,s \wedge \\
 & y = (w/2)\,\grave{}mkbal\grave{}\,t
\end{array}
$$

Conclusion

Pattern ingredients can often be constructed by simple and straightforward calculation

So Let Us Calculate

Conclusion

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